On the Edge Spectrum of Saturation Number for Paths and Stars

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Outline

1 Introduction
   ■ Definitions
   ■ Known Results

2 Edge Spectrum of Paths
   ■ Sketch of the Proof
Definitions

Definition (H-saturated graphs)
A graph $G$ is **H-saturated** if

- $H$ is not a subgraph of $G$,
- for any edge $e \in E(\bar{G})$ the graph $G + e$ contains a subgraph isomorphic to $H$. 
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**Question**

*What is the maximum number of edges in an H-saturated graph on n vertices?*
A classical question in extremal graph theory is:

**Question**

*What is the maximum number of edges in an $H$-saturated graph on $n$ vertices?*

**Definition (Extremal Number)**

The **extremal number** $\text{ex}(n; H)$ defined as

$$\text{ex}(n; H) = \max\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } H\text{-saturated}\}$$

and the set of graphs with extremal number by $\text{Ex}(n; H)$. 
Known Results

Theorem (Mantel, 1907)

\[ \text{ex}(n; K_3) \leq \frac{n^2}{4} \]

and

\[ \text{Ex}(n; K_3) = \{ K_{\frac{n}{2}, \frac{n}{2}} \} \]

The generalization of this theorem to cliques of size \( p \) is
Known Results

Theorem (Turán, 1941)

\[ \text{ex}(n; K_p) \leq \left(1 - \frac{1}{p-1}\right) \frac{n^2}{2} \]

and

\[ \text{Ex}(n; K_p) = \{ T(n, p-1) \} \]

where \( T(n, p-1) \) is called the Turán Graph which is a \((p-1)\)-partite graph with all parts having size as close as possible.
Definitions

One more question.
Definitions

One more question.

Question

What is the minimum number of edges in an $H$-saturated graph on $n$ vertices?
One more question.

**Question**

*What is the minimum number of edges in an H-saturated graph on n vertices?*

**Definition (Saturation Number)**

The **saturation number** \( sat(n; H) \) defined as

\[
sat(n; H) = \min\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } H\text{-saturated}\},
\]

and denote the set of graphs with saturation number by \( Sat(n; H) \).
Definitions

One more question.

**Question**

*What is the minimum number of edges in an $H$-saturated graph on $n$ vertices?*

**Definition (Saturation Number)**

The saturation number $sat(n; H)$ defined as

$$sat(n; H) = \min\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } H\text{-saturated}\},$$

and denote the set of graphs with saturation number by $Sat(n; H)$.

Let $SAT(n; H)$ be the set of all $H$-saturated graphs of order $n$. 
Theorem (Erdős, Hajnal, and Moon, 1964)

For \( n \geq p - 2 \),

\[
\text{sat}(n; K_p) = \binom{p-2}{2} + (n - p + 2)(p - 2)
\]

and

\[
\text{Sat}(n; K_p) = \{K_{p-2} + \overline{K}_{n-p+2}\}
\]

where \( + \) denotes the join operation of graphs.

In particular, \( \text{sat}(n; K_3) = n - 1 \).
Some Known Results for $sat(n; H)$

- $K_{1,k}$, $P_k$ and Matchings studied by Kászonyi and Zs. Tuza, 1986
- $B_p$ studied by Chen, Faudree, Gould, 2008
- $tK_p$ studied by Faudree, Ferrara, Gould, Jacobson, 2009
- Trees studied by J. Faudree, R. Faudree, Gould, Jacobson, 2009
- Cycles: $sat(n; C_n) = \frac{3n+1}{2}$, but exact value of $sat(n; C_k)$ is not known. $sat(n; C_k) = n + \frac{n}{k} + O(k^2 + \frac{n}{k^2})$, by Füredi, Kim, 2011
Theorem (Kászonyi and Zs. Tuza, 1986)

Let \( a_k = \begin{cases} 
3.2^{p-1} - 2, & \text{if } k = 2p \\
4.2^{p-1} - 2, & \text{if } k = 2p + 1.
\end{cases} \)

Then for \( n \geq a_k \) and \( k \geq 6 \),

\[
sat(n; P_k) = n - \left\lfloor \frac{n}{a_k} \right\rfloor \text{ and every graph in } Sat(n; P_k) \text{ consists of a forest with } \left\lfloor \frac{n}{a_k} \right\rfloor \text{ components.}
\]

For \( k = 2p + 2 \), forest consist of binary tree \( T_k \) with root degree 3 and depth \( p \), and for \( k = 2p + 1 \), it is a binary tree with double root both of which have degree 3 and depth \( p \).
Theorem (Kászonyi and Zs. Tuza, 1986)

Let $K_{1,k}$ denote a star on $k + 1$ vertices. Then,

$$\text{sat}(n; K_{1,k}) = \begin{cases} \binom{k}{2} + \binom{n-k}{2} & \text{if } k + 1 \leq n \leq \frac{3k}{2} \\ \left\lfloor n\left(\frac{k-1}{2}\right) - \frac{k^2}{8} \right\rfloor & \text{if } \frac{3k}{2} \leq n \end{cases}$$

and

$$\text{Sat}(n; K_{1,k}) = \begin{cases} \{K_k \cup K_{n-k}\} & \text{if } k + 1 \leq n \leq \frac{3k}{2} \\ \{G' \cup K_{\left\lfloor \frac{k+1}{2}\right\rfloor}\} & \text{if } \frac{3k}{2} \leq n \end{cases}$$

where $G'$ is a $(k - 1)$-regular graph on $n - \left\lfloor \frac{k+1}{2}\right\rfloor$ vertices.
Known Results

Theorem

For \( n \geq k + 1 \), \( \text{ex}(n; K_{1,k}) = \left\lfloor n \frac{k-1}{2} \right\rfloor \), and the extremal graphs are

\[
\text{Ex}(n; K_{1,k}) = \begin{cases} 
(k - 1)\text{-regular graph on } n \text{ vertices} & \text{if } n \text{ is even or } k \text{ is odd} \\
The graph with degree sequence \( k - 1, k - 1, \ldots, k - 1, k - 2 \) & \text{otherwise.}
\end{cases}
\]
Known Results

In 1959, P. Erdős and T. Gallai determined the extremal number $\text{ex}(n; P_k)$ as well as the corresponding extremal graphs. We state the general version of the theorem here by Faudree and Schelp.

**Theorem**

For $n = l(k - 1) + r$,

$$\text{ex}(n; P_k) \leq l\binom{k-1}{2} + \binom{r}{2}$$

with equality if and only if $G$ is either

- (i) $\bigcup_{i=1}^{l} K_{k-1} \cup K_r$ or
- (ii) $\left(\bigcup_{i=1}^{l-t-1} K_{k-1}\right) \cup (K_{(k-2)/2} + \tilde{K}_{(k+2)/2+t(k-1)+r})$ for some $t$, $0 \leq t < l$, when $k$ is even, $l > 0$, and $r = k/2$ or $(k - 2)/2$. 
Known Results

In 77 Kopylov determined $ex(n; P_k)$ for connected graphs. Later on, Balister, Győri, Lehel, and Schelp obtained $ex(n; P_k)$ and also gave $Ex(n; P_k)$.

**Theorem (Balister, Győri, Lehel, and Schelp, 2008)**

If $G$ is connected, then

$$ex(n; P_k) \leq \max\{\binom{k-2}{2} + (n - k + 2), \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k-2}{2} \right\rfloor (n - \left\lceil \frac{k}{2} \right\rceil)\}.$$  

If equality occurs then $G$ is either $G_{n,k,1}$ or $G_{n,k,\lfloor (k-2)/2 \rfloor}$.

Where $G_{n,k,s} = K_s + (K_{k-2s-1} \cup \bar{K}_{n-k+s+1})$, for $k > 2s + 1$. 

Ali Dogan (UofM)  
Edge Spectra for Paths and Stars
What is the edge spectrum for $H$-saturated graphs?

**Definition (The Edge Spectrum for $H$-saturated Graphs)**

The set of all values of $m$, where $\text{sat}(n; H) \leq m \leq \text{ex}(n; H)$, for which there exists an $H$-saturated graph on $n$ vertices and $m$ edges is called the **edge spectrum** for $H$-saturated graphs.
Known Results

Theorem (Barefoot, Casey, Fisher, and Fraughnaugh, 1995)

There is a $K_3$-saturated graph with $n$ vertices and $m$ edges if and only if

$$2n - 5 \leq m \leq \frac{(n-1)^2}{4} + 1 \text{ or } m = k(n - k) \text{ for some positive integer } k.$$
Theorem (Barefoot, Casey, Fisher, and Fraughnaugh, 1995)

There is a $K_3$-saturated graph with $n$ vertices and $m$ edges if and only if
$2n - 5 \leq m \leq \frac{(n-1)^2}{4} + 1$ or $m = k(n - k)$ for some positive integer $k$.

Theorem (Kinnari, Faudree, Gould and Sidorowicz, 2013)

There is a $K_p$-saturated graph with $n$ vertices and $m$ edges if and only if either

$$(p - 1)\left(n - \frac{p}{2}\right) - 2 \leq m \leq \frac{(p - 2)n^2 - 2n + r(r + 2) - r(p - 1)}{2(p - 1)} + 1$$

or $m = |E(G)|$ for some complete $(p - 1)$-partite graph $G$ on $n$ vertices.

Recall $sat(n; K_p) = \left(\frac{p-2}{2}\right) + (n - p + 2)(p - 2)$ by (EHM, 1964) and
$ex(n; K_p) \leq (1 - \frac{1}{p-1})\frac{n^2}{2}$ by (Turan, 1941).
Theorem (Gould, Tang, Wei, Zhang, 2012)

There are $P_5$-saturated graphs with $n$ vertices and $m$ edges provided $sat(n; P_5) \leq m \leq ex(n; P_5)$, except in the cases

\[
m \in \begin{cases} 
\left\{ \frac{3n-5}{2} \right\} & \text{if } n \equiv 3 \pmod{4}, \\
\left\{ \frac{3n}{2} - 3, \frac{3n}{2} - 2, \frac{3n}{2} - 1 \right\} & \text{if } n \equiv 0 \pmod{4}.
\end{cases}
\]

Note that $sat(n; P_5) = \frac{5n-4}{6}$ by (KT, 1986) and $ex(n; P_5) = \frac{3n}{2}$ by (EG, 1959).
Known Results

**Theorem (Gould, Tang, Wei, Zhang, 2012)**

For $n \geq 10$ and $(n, m) \neq (11, 14)$, there are $P_6$-saturated graphs with $n$ vertices and $m$ edges provided $\text{sat}(n; P_6) \leq m \leq \text{ex}(n; P_6)$, except in the cases

$$m \in \begin{cases} 
\{2n - 4, 2n - 2, 2n - 1\} & \text{if } n \equiv 0 \pmod{5}, \\
\{2n - 4\} & \text{if } n \equiv 2, 4 \pmod{5}.
\end{cases}$$

Note that $\text{sat}(n; P_6) = \frac{9n}{10}$ by (KT, 1986) and $\text{ex}(n; P_6) = 2n$ by (EG, 1959).
Our Main Results

Theorem

Let $\epsilon > 0$, and let $k$ and $n$ be integers with $k \geq k_0(\epsilon)$ and $n \geq a_k$, where $a_k$ is defined previously. Then for any integer $m$ such that $\text{sat}(n; P_k) \leq m \leq \text{ex}(n; P_k) - (\sqrt{2} + \epsilon)k^{3/2}$ there exists a $P_k$-saturated graph on $n$ vertices with $m$ edges.
Our Main Results

Theorem

Let $\epsilon > 0$, and let $k$ and $n$ be integers with $k \geq k_0(\epsilon)$ and $n \geq a_k$, where $a_k$ is defined previously. Then for any integer $m$ such that $\text{sat}(n; P_k) \leq m \leq \text{ex}(n; P_k) - (\sqrt{2} + \epsilon)k^{3/2}$ there exists a $P_k$-saturated graph on $n$ vertices with $m$ edges.

We also show that $\text{ex}(n; P_k) - (\sqrt{2} + o(1))k^{3/2}$ is the best possible upper bound, up to the constant $\sqrt{2}$. More precisely, we show that for each sufficiently large $k$ there exists an infinite sequence of $n$ and $m$ with $\text{sat}(n; P_k) \leq m \leq \text{ex}(n; P_k) - \epsilon k^{3/2}$, and no $P_k$-saturated graph exists with $n$ vertices and $m$ edges.
Our Main Results...

Theorem

Let $n$ and $k$ be two integers such that $n \geq k \geq 1$. Then for any integer $m$ such that $\text{sat}(n; K_{1,k}) \leq m \leq \text{ex}(n; K_{1,k})$ there is a $K_{1,k}$-saturated graph on $n$ vertices with $m$ edges.
We use the following Lemma to get the top part of the edge spectrum for paths...

**Lemma**

Let $f(n)$ be the largest integer such that every integer between 0 and $f(n)$ can be represented as $\sum_{i \geq 1} (r_i)$ with $\sum_{i \geq 1} r_i = n$ and integers $r_i \geq 0$. Then $f(n) \geq \frac{1}{2} (n - 2\sqrt{n})^2$ for $n \geq 2$. 
Sketch of the Proof

Part 1. Let \( n = l(k - 1) + r, 0 \leq r < k - 1 \). We will deal with the cases when \( r \) is large and \( r \) is small separately.

Case 1.1: \( \binom{r}{2} \geq k - 2 \).

Let

\[
G_0 = \left( \bigcup_{i=1}^{l-s} K_{k-1} \right) \cup \left( \bigcup_{i=1}^{s-1} K_{k-2} \right) \cup H \cup K_r,
\]

where \( H = K_1 + (K_{k-3} \cup \bar{K}_s) \).

Then \( |E(G_0)| = e - s(k - 3) \), where \( e = \text{ex}(n; P_k) \).
Case 1.2: \( \binom{r}{2} < k - 2 \). In this case we let

\[
G_1 = \left( \bigcup_{i=1}^{l-s} K_{k-1} \right) \cup \left( \bigcup_{i=1}^{s-1} K_{k-2} \right) \cup H_{r+s},
\]

where \( H_{r+s} = K_1 + (K_{k-3} \cup \bar{K}_{s+r}) \). Note that this graph is saturated provided \( r + s \geq 2 \), and \( |E(G_1)| = e - s(k - 3) - a \), where \( e = \text{ex}(n; P_k) \) and \( a = \binom{r}{2} - r \).
Sketch of the Proof

The following claim tells us that by moving vertices from some cliques to other cliques in $G_0$ (or in $G_1$) we gain some edges.

Claim:
Replacing the cliques $K_{k-2}$ by $K_{k-1-ri}$ in graph $G_0$ so that their total order remains constant always gains $\sum(ri^2)$ edges, where $ri \geq 0$ are such that $\sum_i r_i = |I|$ is the original number of $K_{k-2}$ cliques.

Proof of Claim:
Replacing $K_{k-2}$ by $K_{k-1-ri}$ changes the number of edges by $(k-1-ri^2) - (k-2)^2 = \frac{1}{2}((k-1-ri)(k-2-ri) - (k-2)(k-3)) = \frac{1}{2}(ri^2 - (2k-3)ri + 2(k-2))$.
Summing over $i$ and noting that $\sum (1-ri) = 0$ gives the result.
The following claim tells us that by moving vertices from some cliques to other cliques in $G_0$ (or in $G_1$) we gain some edges.

**Claim:** Replacing the cliques $K_{k-2}$ by $K_{k-1-r_i}$ in graph $G_0$ so that their total order remains constant always gains $\sum \binom{r_i}{2}$ edges, where $r_i \geq 0$ are such that $\sum_{i \in I} r_i = |I|$ is the original number of $K_{k-2}$ cliques.
The following claim tells us that by moving vertices from some cliques to other cliques in $G_0$ (or in $G_1$) we gain some edges.

**Claim:** Replacing the cliques $K_{k-2}$ by $K_{k-1-r_i}$ in graph $G_0$ so that their total order remains constant always gains $\sum \left(\begin{array}{c} r_i \\ 2 \end{array}\right)$ edges, where $r_i \geq 0$ are such that $\sum_{i \in I} r_i = |I|$ is the original number of $K_{k-2}$ cliques.

**Proof of Claim:** Replacing $K_{k-2}$ by $K_{k-1-r_i}$ changes the number of edges by

$$
\left(\begin{array}{c} k-1-r_i \\ 2 \end{array}\right) - \left(\begin{array}{c} k-2 \\ 2 \end{array}\right) = \frac{1}{2} \left( (k-1-r_i)(k-2-r_i) - (k-2)(k-3) \right) \\
= \frac{1}{2} \left( r_i^2 - (2k-3)r_i + 2(k-2) \right) = (k-2)(1-r_i) + \left(\begin{array}{c} r_i \\ 2 \end{array}\right).
$$

Summing over $i$ and noting that $\sum (1-r_i) = 0$ gives the result.
**Part 2:** Define *Forming a Pendant Triangle* at a vertex $x$ as follow: remove two vertices from the $s$ pendant vertices and form a triangle whose vertices are the vertex $x$ and two removed vertices. By Forming a Pendant Triangle at a vertex $x$ we gain exactly 1 edge, and the resulting graph is still $P_k$-saturated.

*Figure:* Forming a Pendant Triangle at $x$
Part 3: We start with a smallest saturated graph $G$, which is a forest consisting of almost binary trees $T_k$, with $|E(G)| = sat(n; P_k) = n - \left\lfloor \frac{n}{a_k} \right\rfloor$. By forming pendant triangles in a tree component many times, we cover the bottom part of the spectrum.

Figure: Forming a Pendant Triangle on the bottom level
There is a gap near $\text{ex}(n : P_k)$

**Corollary**

Let $k$ be sufficiently large, and let $n = (k - 1)l$. Then there is an integer $\beta_0 \sim k^{3/2}/\sqrt{24}$ such that there is no $P_k$-saturated graph of size $\text{ex}(n; P_k) - \beta_0$. 
The End
Thank You!