Solvable regular covering projections of graphs

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3rd annual Mississippi Discrete Mathematics Workshop
Mississippi State University

November 15-16, 2014
Graph covers

Regular coverings of connected graphs

A surjective mapping \( p: \tilde{X} \to X \) such that fibres \( p^{-1}(v) \) are orbits of a semiregular subgroup \( C_T \).

Construction/reconstruction by a regular voltage function \( \zeta: X \to \Gamma \sim C_T \).
Regular coverings of connected graphs

a surjective mapping $p: \tilde{X} \to X$ s.t.

fibres $p^{-1}(v) = \text{orbits of a semiregular subgroup } CT_p \leq \text{Aut}(\tilde{X})$
Regular coverings of connected graphs

a surjective mapping \( p : \tilde{X} \rightarrow X \) s.t.

fibres \( p^{-1}(v) = \) orbits of a semiregular subgroup \( CT_p \leq \text{Aut}(\tilde{X}) \)

Construction/reconstruction

by a regular voltage function \( \zeta : X \rightarrow \Gamma \cong CT_p \)
Symmetries of covering graph vs. base graph

Lifting automorphisms along regular covering projections

\[ \tilde{X} \xrightarrow{\tilde{g}} X \]

all elements of \( G \leq \text{Aut}(X) \) lift \( G \)-admissible regular cover

Applications:
- classification of particular classes of graphs and maps on surfaces,
- counting the number of graphs in certain families,
- constructing infinite families or produce catalogues of graphs with prescribed degree of symmetry up to a certain reasonable size
Lifting automorphisms along regular covering projections

\[ \tilde{X} \xrightarrow{\tilde{g}} \tilde{X} \]
\[ p \downarrow \quad \quad \quad \downarrow p \]
\[ X \xrightarrow{g} X \]
Lifting automorphisms along regular covering projections

\[ \tilde{X} \xrightarrow{\tilde{g}} \hat{X} \]
\[ p \downarrow \quad \quad \downarrow p \]
\[ X \xrightarrow{g} X \]

_all elements of \( G \leq \text{Aut}(X) \) lift_

\( G \)-admissible regular cover
Lifting automorphisms along regular covering projections

\[
\begin{align*}
\tilde{X} & \xrightarrow{\tilde{g}} \tilde{X} \\
p & \downarrow p \\
X & \xrightarrow{g} X
\end{align*}
\]

all elements of \( G \leq \text{Aut}(X) \) lift

\( G \)-admissible regular cover

Applications

classification of particular classes of graphs and maps on surfaces,
counting the number of graphs in certain families,
constructing infinite families or produce catalogues of graphs
with prescribed degree of symmetry up to a certain reasonable size
The structure of the lifted group

\[ \tilde{\mathcal{G}} \text{ is a group extension of } \mathcal{C} \mathcal{T}_p \text{ by } \mathcal{C} \mathcal{T}_p \triangleright \tilde{\mathcal{G}} \]

\[ \tilde{\mathcal{G}} / \mathcal{C} \mathcal{T}_p \cong \mathcal{G} / \mathcal{G}_{12} \]
The structure of the lifted group

\[ \tilde{G} \] is a group extension of \( CT_p \) by \( G \)

\[ CT_p \triangleleft \tilde{G} \] and \( \tilde{G}/CT_p \cong G \)
Universal covering projection

\[ \text{Universal property for every } p : \tilde{X} \to X \text{ there exists a unique } q : T \to \tilde{X} \text{ s.t.} \]

\[ T \quad \tilde{X} \quad X \]

\[ p \quad q \quad p^* \]

\[ T \sim \pi(X, u_0) \]
covering projection \( p^* : \mathcal{T} \rightarrow X \)

where \( \mathcal{T} \) is a tree
covering projection $p^* : \mathcal{T} \to X$

where $\mathcal{T}$ is a tree

Universal property

for every $p : \tilde{X} \to X$ there exists a unique $q : \mathcal{T} \to \tilde{X}$ s.t.

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{q} & \tilde{X} \\
\downarrow p^* & & \downarrow p \\
X & \xleftarrow{p} & \tilde{X}
\end{array}
\]
covering projection \( p^* : T \rightarrow X \)

where \( T \) is a tree

**Universal property**

for every \( p : \tilde{X} \rightarrow X \) there exists a unique \( q : T \rightarrow \tilde{X} \) s.t.

\[
\begin{array}{ccc}
T & \xrightarrow{q} & \tilde{X} \\
\downarrow{p^*} & & \downarrow{p} \\
X & & \end{array}
\]

**Additional properties**

\( p^* \) is \( G \)-admissible for any \( G \leq \text{Aut}(X) \);

\( CT_{p^*} \cong \pi(X, u_0) \)
$N \triangleleft C\Gamma p^*$ $\iff$ regular covering projections
For a normal subgroup $N$ of $\text{CT}_p^*$

\[
\begin{array}{c}
\text{T} \\
p^*
\end{array}
\begin{array}{c}
\downarrow \\
\rightarrow \\
p
\end{array}
\begin{array}{c}
\tilde{X} \\
X
\end{array}
\begin{array}{c}
\downarrow \\
\rightarrow \\
q
\end{array}
\]

$\text{CT}_p \cong \text{CT}_p^*/N$;
$\text{CT}_q \cong N \cong \pi(\tilde{X}, \tilde{u}_0)$
Which $N \triangleleft CT_{p^*}$ give rise to $G$-admissibility?
Which $N \triangleleft CT_{p^*}$ give rise to $G$-admissibility?

\[ \tilde{T} \xrightarrow{\text{q}} \tilde{X} \]
\[ \tilde{X} \xrightarrow{\text{p}} X \]

Suppose $p$ is $G$-admissible. Then $\tilde{q}$ is $\tilde{G}$-admissible, and

\[ \tilde{G} \cong \text{the lifted group of } G \text{ along } p \]

such that

\[ N \triangleleft CT_{p^*} \leftrightarrow G \text{-admissible regular coverings} \]
Which \( N \triangleleft CT_{p^*} \) give rise to \( G \)-admissibility?

\[ G^* = \text{the lifted group of } G \text{ along } p_{\zeta^*} (CT_{p^*} \triangleleft G^* \text{ and } G^*/CT_{p^*} \cong G) \]

\[ N \triangleleft CT_{p^*} \triangleleft G^* \]
Which $N \triangleleft C T_{p^*}$ give rise to $G$-admissibility?

$G^* = \text{the lifted group of } G \text{ along } p_{\zeta^*} \left( C T_{p^*} \triangleleft G^* \text{ and } G^*/C T_{p^*} \cong G \right)$

$N \triangleleft C T_{p^*} \triangleleft G^*$

**Suppose $p$ is $G$-admissible**

$
\tilde{G} = \text{the lifted group of } G \text{ along } p
$
Which $N \triangleleft CT_{\pi}^*$ give rise to $G$-admissibility?

$G^* = \text{the lifted group of } G \text{ along } p_{\zeta^*}(CT_{\pi}^* \triangleleft G^* \text{ and } G^*/CT_{\pi}^* \cong G)$

$N \triangleleft CT_{\pi}^* \triangleleft G^*$

Suppose $p$ is $G$-admissible

$	ilde{G} = \text{the lifted group of } G \text{ along } p$

Then $q$ is $\tilde{G}$ – admissible
Which \( N \triangleleft \text{CT}_{p^*} \) give rise to \( G \)-admissibility?

\[
\begin{array}{c}
\tau \\
\downarrow q \\
p^* \\
\downarrow p
\end{array}
\begin{array}{c}
\tilde{X} \\
\downarrow p
\end{array}
\begin{array}{c}
X
\end{array}
\]

\[ G^* = \text{the lifted group of } G \text{ along } p \zeta^* (\text{CT}_{p^*} \triangleleft G^* \text{ and } G^*/\text{CT}_{p^*} \cong G) \]

\[ N \triangleleft \text{CT}_{p^*} \triangleleft G^* \]

**Suppose** \( p \) is \( G \)-admissible

\[ \tilde{G} = \text{the lifted group of } G \text{ along } p \]

**Then** \( q \) is \( \tilde{G} \) – admissible

\[
\begin{array}{c}
G^* \\
\downarrow \\
\tilde{G} \\
\downarrow \\
G
\end{array}
\]

\[ N \triangleleft G^* \text{ s.t. } N \leq \text{CT}_{p^*} \leftrightarrow G \text{-admissible regular coverings} \]
Concrete computations?

Finding a presentation of $G^*$ reconstruct $p^*$ in terms of voltages, use general recipe for constructing a presentation of $G^*$ as an extension of $CT_p^*$ by $G$, find $G = \langle g_1, \ldots, g_n | r_1(g_1, \ldots, g_n), \ldots, r_m(g_1, \ldots, g_n) \rangle$, use formula for evaluating lifts of automorphisms
Finding a presentation of $G^*$

reconstruct $p^*$ in terms of voltages,

use general recipe for constructing a presentation of $G^*$ as an extension of $CT_{p^*}$ by $G$,

$$G = \langle g_1, \ldots, g_n \mid r_1(g_1, \ldots, g_n), \ldots, r_m(g_1, \ldots, g_n) \rangle,$$

use formula for evaluating lifts of automorphisms
$G$-admissible solvable regular covering projections

[13x257]G-admissible solvable regular covering projections

Up to a prescribed order $n$ of the respective covering graphs find all $\mathcal{N} \triangleleft G \ast$ contained in $\mathcal{C} \ast \mathcal{T}$ with $\mathcal{C} \ast \mathcal{T} / \mathcal{N}$ solvable of order at most $n$

The basic idea in a solvable $\mathcal{C} \ast \mathcal{T} / \mathcal{N}$ there exists a normal elementary abelian subgroup $K / \mathcal{N}$; if $K$ is known, $\mathcal{N}$ can be found by considering $H \triangleleft G \ast$ with $H \leq K$ and $K / H$ elementary abelian
Up to a prescribed order $n$ of the respective covering graphs
find all $N \triangleleft G^*$ contained in $CT_{p^*}$ with $CT_{p^*}/N$ solvable of order at most $n$
Up to a prescribed order \( n \) of the respective covering graphs
find all \( N \triangleleft G^* \) contained in \( C T_{p^*} \) with \( C T_{p^*}/N \) solvable of order at most \( n \)

The basic idea

in a solvable \( C T_{p^*}/N \) there exists a normal elementary abelian subgroup \( K/N \); if \( K \) is known, \( N \) can be found by considering \( H \triangleleft G^* \) with \( H \leq K \) and \( K/H \) elementary abelian
An algorithm

Computing normal subgroups with solvable factor

Input: a finitely presented group \( G \), a normal subgroup \( F \) of \( G \), given by words in the generators of \( G \) that generates \( F \), an integer \( n > 0 \)

Output: the set \( N \) of all normal subgroups \( N \) of \( G \) contained in \( H \) with \( F / N \) solvable of order at most \( n \)

1: \[ N = \{ F \} \]
2: \[ \text{Processed} = \emptyset \]
3: \[ \text{while } N \setminus \text{Processed} \neq \emptyset \text{ do} \]
4: \[ \text{Choose } K \in N \setminus \text{Processed} \text{ and insert } K \text{ in } \text{Processed}; \]
5: \[ \text{foreach prime } p \text{ with } p \mid F : K \mid \leq n \text{ do} \]
6: \[ \text{Let } M = K / [K, K] \]
7: \[ \text{K}_p \text{ with } f : K \to M \text{ the natural epimorphisms}; \]
8: \[ \text{Turn } M \text{ into } \mathbb{Z}[G^*/K] \text{-module}; \]
9: \[ \text{Find the set } S \text{ of all maximal } \mathbb{Z}[G^*/K] \text{-submodules of } M \text{ whose codimension } d \text{ satisfies } p^d \mid F : K \mid \leq n \];
10: \[ \text{foreach } S \in S \text{ do} \]
11: \[ \text{Let } L = f^{-1}(S); \]
12: \[ \text{if } L \text{ is not equal to any of subgroups in } N \text{ then} \]
13: \[ \text{Insert } L \text{ into } N; \]
14: \[ \text{return } N; \]
An algorithm

Computing normal subgroups with solvable factor

**Input:** a finitely presented group $G^*$, a normal subgroup $F$ of $G^*$ given by words in the generators of $G^*$ that generates $F$, an integer $n > 0$

**Output:** the set $\mathcal{N}$ of all normal subgroups $N$ of $G$ contained in $H$ with $F/N$ solvable of order at most $n$

1. Set $\mathcal{N} = \{F\}$ and set $\text{Processed} = \emptyset$;
2. while $\mathcal{N} \setminus \text{Processed} \neq \emptyset$ do
3. Choose $K \in \mathcal{N} \setminus \text{Processed}$ and insert $K$ in $\text{Processed}$;
4. foreach prime $p$ with $p|F : K| \leq n$ do
6. Turn $M$ into $\mathbb{Z}_p[G^*/K]$-module;
7. Find the set $S$ of all maximal $\mathbb{Z}_p[G^*/K]$-submodules of $M$ whose codimension $d$ satisfies $p^d|F : K| \leq n$;
8. foreach $S \in S$ do
9. Let $L = f^{-1}(S)$;
10. if $L$ is not equal to any of subgroups in $\mathcal{N}$ then
11. Insert $L$ into $\mathcal{N}$;
12. return $\mathcal{N}$;
$G^*$

$F$
\[ G^* \]

\[ F \]

\[ [F, F]F^p \]
$G^*$

$F$

$[F, F]F^p$

$M = F/[F, F]F^p$
\[ M = F/[F, F]^p \]

\[ \mathbb{Z}_p[G^*/F] - \text{module} \]
\[ M = F/[F, F]F^p \]

\[ \mathbb{Z}_p[G^* / F] \text{– module} \]
\[ M = F/[F, F]F^p \]

\[ \mathbb{Z}_p[G^*/F] \text{–module} \]

\[ S \]
Thank you!