Multiplicative Zagreb indices of $k$-trees

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Directed by Bing Wei

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- Introduction
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  - $k$-trees
  - Zagreb Index and Multiplicative Zagreb Index
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Our results
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- Introduction
  - $k$-trees
  - Zagreb Index and Multiplicative Zagreb Index
- Our results
- Main proofs
Definition (Beineke and Pippert 1969)

The **k-tree**, denoted by $T_n^k$, for positive integers $n, k$ with $n \geq k$, is defined recursively as follows:

The smallest $k$-tree is the $k$-clique $K_k$. If $G$ is a $k$-tree with $n \geq k$ vertices and a new vertex $v$ of degree $k$ is added and joined to the vertices of a $k$-clique in $G$, then the obtained graph is a $k$-tree with $n + 1$ vertices.
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**Example (Building a 2-tree)**

- Start with a 1-vertex graph.
- Add a new vertex of degree 2, and join it to the existing vertex.

Diagram:

1. **Initial Graph**: A single vertex.
2. **Updated Graph**: The initial vertex, with a new vertex added and joined to it.
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**Example (Building a 2-tree)**

![Diagram of a 2-tree](image-url)
Definition (Beineke and Pippert 1969)

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**Example (Building a 2-tree)**

[Diagram showing a 2-tree construction with labeled vertices and edges]
Definition

A vertex $v \in V(T_n^k)$ is called a $k$-simplicial vertex if $v$ is a vertex of degree $k$ whose neighbors form a $k$-clique of $T_n^k$. 

In the following 2-tree, 5, 6, 7 are 2-simplicial vertices.
A vertex $v \in V(T_k^n)$ is called a $k$-simplicial vertex if $v$ is a vertex of degree $k$ whose neighbors form a $k$-clique of $T_k^n$.

In the following 2-tree, 5, 6, 7 are 2-simplicial vertices.
Let $S_1(T_n^k)$ be the set of all simplicial vertices of $T_n^k$, for $n \geq k + 2$, and set $S_1(K_k) = \phi$, $S_1(K_{k+1}) = \{v\}$, where $v$ is any vertex of $K_{k+1}$. 
Let $S_1(T_n^k)$ be the set of all simplicial vertices of $T_n^k$, for $n \geq k + 2$, and set $S_1(K_k) = \emptyset$, $S_1(K_{k+1}) = \{v\}$, where $v$ is any vertex of $K_{k+1}$.

Let $G = G_0$, $G_i = G_{i-1} - v_i$, where $v_i$ is a simplicial vertex of $G_{i-1}$, then $\{v_1, v_2...v_n\}$ is called a simplicial elimination ordering of the $n$-vertex graph $G$. 
The \textit{k-path}, denoted by $P^k_n$, for positive integers $n$, $k$ with $n \geq k$, is defined as follows:
Starting with a $k$-clique $G[\{v_1, v_2...v_k\}]$. For $i \in [k + 1, n]$, the vertex $v_i$ is adjacent to vertices $\{v_{i-1}, v_{i-2}...v_{i-k}\}$ only.
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Example (Building a 2-path)

1

2
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The \textbf{k-path}, denoted by $P_n^k$, for positive integers $n, k$ with $n \geq k$, is defined as follows:
Starting with a $k$-clique $G[\{v_1, v_2, \ldots, v_k\}]$. For $i \in [k + 1, n]$, the vertex $v_i$ is adjacent to vertices $\{v_{i-1}, v_{i-2}, \ldots, v_{i-k}\}$ only.

\begin{center}
\textbf{Example (Building a 2-path)}
\end{center}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{example_graph}
\end{figure}
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The *k-path*, denoted by $P_n^k$, for positive integers $n, k$ with $n \geq k$, is defined as follows:
Starting with a $k$-clique $G[\{v_1, v_2, \ldots, v_k\}]$. For $i \in [k + 1, n]$, the vertex $v_i$ is adjacent to vertices $\{v_{i-1}, v_{i-2}, \ldots, v_{i-k}\}$ only.

Example (Building a 2-path)
The \textit{k-star}, denoted by $S_{k,n-k}$, for positive integers $n, k$ with $n \geq k$, is defined as follows:

Starting with a $k$-clique $G[\{v_1, v_2...v_k\}]$ and an independent set $S$ with $|S| = n - k$. For $i \in [k + 1, n]$, the vertex $v_i$ is adjacent to vertices $\{v_1, v_2...v_k\}$ only.
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**Example (Building a 2-star)**

![Diagram of a 2-star with vertices 1 and 2 connected]

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Multiplicative Zagreb indices of $k$-trees
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Example (Building a 2-star)

![Diagram of a 2-star with vertices 1, 2, 3, 4, 5, and edges connecting them accordingly.]
The $k$-star, denoted by $S_{k,n-k}$, for positive integers $n, k$ with $n \geq k$, is defined as follows: Starting with a $k$-clique $G[\{v_1, v_2...v_k\}]$ and an independent set $S$ with $|S| = n - k$. For $i \in [k + 1, n]$, the vertex $v_i$ is adjacent to vertices $\{v_1, v_2...v_k\}$ only.
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Example (Building a 2-star)

[Diagram of a 2-star with vertices labeled 1 to 7 and edges connecting them as described in the definition.]

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Multiplicative Zagreb indices of $k$-trees
Hyper pendent edge

Definition

If \( w(G - S) \leq 2 \) for any \( k \)-clique \( G[S] \) of \( T_n^k \), we say \( T_n^k \) is a hyper pendent edge; If there exists a \( k \)-clique \( G[S] \) with \( w(G - S) \geq 3 \), let \( C \) be a component of \( T_n^k - S \) and contain a unique vertex belonging to \( S_1(G) \), then we say that \( G[V(S) \cup V(C)] \) is a hyper pendent edge of \( T_n^k \), denoted by \( \mathcal{P} \).
Hyper pendent edge

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If \( w(G - S) \leq 2 \) for any \( k \)-clique \( G[S] \) of \( T_n^k \), we say \( T_n^k \) is a hyper pendent edge; If there exists a \( k \)-clique \( G[S] \) with \( w(G - S) \geq 3 \), let \( C \) be a component of \( T_n^k - S \) and contain a unique vertex belonging to \( S_1(G) \), then we say that \( G[V(S) \cup V(C)] \) is a hyper pendent edge of \( T_n^k \), denoted by \( P \).

- 2-tree: \( S = \{1, 2\} \), \( C = \{3, 4, 5, 6, 7\} \)
Zagreb Indices

**Definition**

The first and second **Zagreb indices** of the graph $G = (V, E)$ are defined as

$$M_1 = \sum_{v \in V(G)} d(v)^2; \quad M_2 = \sum_{uv \in E(G)} d(u)d(v).$$
Zagreb Indices

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The first and second **Zagreb indices** of the graph $G = (V, E)$ are defined as

\[ M_1 = \sum_{v \in V(G)} d(v)^2; \quad M_2 = \sum_{uv \in E(G)} d(u)d(v). \]

Let $P_n$, $S_n$ be the path and star on $n$ vertices, respectively, then

\[ M_1(P_n) = 4n - 6, \quad M_1(S_n) = n^2 - n; \]
\[ M_2(P_n) = 4n - 8, \quad M_2(S_n) = n^2 - 2n + 1. \]
Theorem (Das and Gutman 2004)

Let $T$ be any tree on $n$ vertices, then

$$M_1(P_n) \leq M_1(T) \leq M_1(S_n)$$
$$M_2(P_n) \leq M_2(T) \leq M_2(S_n)$$

the left-side and the right-side equalities are reached if and only if $T \cong P_n$ and $T \cong S_n$, respectively.
**Theorem (Das and Gutman 2004)**

Let $T$ be any tree on $n$ vertices, then

\[
M_1(P_n) \leq M_1(T) \leq M_1(S_n)
\]

\[
M_2(P_n) \leq M_2(T) \leq M_2(S_n)
\]

The left-side and the right-side equalities are reached if and only if $T \cong P_n$ and $T \cong S_n$, respectively.

**Theorem (Estes and Wei 2012)**

Let $T^k_n$ be any $k$-tree on $n$ vertices, then

\[
M_1(P^k_n) \leq M_1(T^k_n) \leq M_1(S_{k,n-k})
\]

\[
M_2(P^k_n) \leq M_2(T^k_n) \leq M_2(S_{k,n-k})
\]

The left-side and the right-side equalities are reached if and only if $T^k_n \cong P^k_n$ and $T^k_n \cong S_{k,n-k}$, respectively.
The first and second **Multiplicative Zagreb indices** of the graph $G = (V, E)$ are defined as

\[
\prod_1(G) = \prod_{v \in V(G)} d(v)^2; \quad \prod_2(G) = \prod_{uv \in E(G)} d(u)d(v).
\]
**Definition (Todeschini, Ballabio, Consonni 2010)**

The first and second **Multiplicative Zagreb indices** of the graph \( G = (V, E) \) are defined as

\[
\prod_1(G) = \prod_{v \in V(G)} d(v)^2; \quad \prod_2(G) = \prod_{uv \in E(G)} d(u)d(v).
\]

---

**Theorem (Gutman 2011)**

*Let \( n \geq 5 \) and \( T_n \) be any tree with \( n \) vertices, then*

\[
\prod_1(S_n) \leq \prod_1(T_n) \leq \prod_1(P_n);
\]

\[
\prod_2(P_n) \leq \prod_2(T_n) \leq \prod_2(S_n).
\]
Definition

The first **generalized** and second **Multiplicative Zagreb indices** of graph $G = (V, E)$ are defined as follows: for any real number $c > 0$,

\[
\prod_{1,c}(G) = \prod_{v \in V(G)} d(v)^c; \\
\prod_2(G) = \prod_{uv \in E(G)} d(u)d(v) = \prod_{v \in V(G)} d(v)^{d(v)}. 
\]
Our results

**Theorem**

Let $T_k^n$ be a $k$-tree on $n \geq k$ vertices, then

\[(1) \prod_{1,c} (S_{k,n-k}) \leq \prod_{1,c} (T_k^n) \leq \prod_{1,c} (P_k^n)\]

\[(2) \prod_{2} (P_k^n) \leq \prod_{2} (T_k^n) \leq \prod_{2} (S_{k,n-k})\]

For (1), the left-side and the right-side equalities are reached if and only if $T_k^n \cong S_{k,n-k}$ and $T_k^n \cong P_k^n$, respectively; For (2), the left-side and the right-side equalities are reached if and only if $T_k^n \cong P_k^n$ and $T_k^n \cong S_{k,n-k}$, respectively.
The function \( f(x) = \frac{x}{x + m} \) is strictly increasing for \( x \in [0, \infty) \), where \( m \) is a positive integer.

The function \( g(x) = \frac{x^x}{(x + m)^{x+m}} \) is strictly decreasing for \( x \in [0, \infty) \), where \( m \) is a positive integer.
We first show that $\prod_{1,c}(T_n^k) \geq \prod_{1,c}(S_{k,n-k})$,
$\prod_{2}(T_n^k) \leq \prod_{2}(S_{k,n-k})$, it suffices to prove the following lemma.

**Lemma (1)**

For any $k$-tree $G \not\cong S_{k,n-k}$, let $u \in S_2$, $N(u) \cap S_1 = \{v_1, v_2...v_s\}$, where $s \geq 1$ is an integer, then

(i) For any $i$ with $1 \leq i \leq s$, there exists a vertex $v \in N(u) - \{v_1, v_2...v_s\}$ of degree at least $k$ in $G[V(G) - \{v_1, v_2...v_s\}]$ such that $vv_i \not\in E(G)$.

(ii) There exists a $k$-tree $G^*$ such that $\prod_{1,c}(G^*) < \prod_{1,c}(G)$ and $\prod_{2}(G^*) > \prod_{2}(G)$.
Proof of (i)

Let $G' = G \setminus \{v_1, v_2, \ldots, v_s\}$ and $S = N(u) \setminus \{v_1, v_2, \ldots, v_s\}$, we obtain that $d_{G'}(u) = |S| = k$ and $G[S]$ is a $k$-clique by $u \in S$. Since $G \not\sim S_k n$, $d_{G'}(v) \geq k$ for all $v \in S$. And by the facts that $N(v_i) \subseteq (N(u) \setminus \{v_1, v_2, \ldots, v_s\}) \cup \{u\}$ with $|N(v_i)| = k$ and $|(N(u) \setminus \{v_1, v_2, \ldots, v_s\}) \cup \{u\}| = k + 1$, we have that for any $i \in [1, s]$ there exists a vertex $v \in S$ such that $v v_i / \in E(G)$. 

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Multiplicative Zagreb indices of $k$-trees
Proof of (i)

Let $G' = G[V(G) - \{v_1, v_2, ..., v_s\}]$ and $S = N(u) - \{v_1, v_2, ..., v_s\}$, we obtain that $d_{G'}(u) = |S| = k$ and $G[S]$ is a $k$-clique by $u \in S_2$. Since $G \not\cong S_n^k$, $d_{G'}(v) \geq k$ for all $v \in S$. And by the facts that $N(v_i) \subseteq (N(u) - \{v_1, v_2, ..., v_s\}) \cup \{u\}$ with $|N(v_i)| = k$ and $|((N(u) - \{v_1, v_2, ..., v_s\}) \cup \{u\})| = k + 1$, we have that for any $i \in [1, s]$ there exists a vertex $v \in S$ such that $vv_i \notin E(G)$.
Proof of (ii)

Choose $v_1$, there is a vertex $v \in S$ with $d_{G'}(v) \geq k$. If $d_{G'}(v) = k$, $G'$ is a $k + 1$-clique.

Let $x \in S$ be the vertex such that $d(x) = \min_{v \in S} \{d(v)\}$, and let $v_t x \in E(G)$, $v_t y \notin E(G)$ for some $t \in [1, s]$ and $y \in S$, that is, $d(x) - 1 < d(y)$. Denote $G_0 = G[V(G) - \{x, y\}]$.

Construct a new graph $G^*$ such that $V(G^*) = V(G)$, and $E(G^*) = E(G) - \{v_t x\} + \{v_t y\}$. 

\[ \text{Proof of (ii)} \]

\[
\begin{align*}
\text{Choose } v_1, \text{ there is a vertex } v \in S \text{ with } d_{G'}(v) \geq k. \text{ If } \\
d_{G'}(v) = k, \text{ } G' \text{ is a } k + 1\text{-clique.} \\
\text{Let } x \in S \text{ be the vertex such that } d(x) = \min_{v \in S} \{d(v)\}, \text{ and } \\
\text{let } v_t x \in E(G), \text{ } v_t y \notin E(G) \text{ for some } t \in [1, s] \text{ and } y \in S, \text{ that is, } \\
d(x) - 1 < d(y). \text{ Denote } G_0 = G[V(G) - \{x, y\}]. \\
\text{Construct a new graph } G^* \text{ such that } V(G^*) = V(G), \text{ and } \\
E(G^*) = E(G) - \{v_t x\} + \{v_t y\}. 
\end{align*}
\]
Proof of (ii)

Choose $v_1$, there is a vertex $v \in S$ with $d_{G'}(v) \geq k$. If $d_{G'}(v) = k$, $G'$ is a $k + 1$-clique.

Let $x \in S$ be the vertex such that $d(x) = \min_{v \in S} \{d(v)\}$, and let $v_t x \in E(G)$, $v_t y \notin E(G)$ for some $t \in [1, s]$ and $y \in S$, that is, $d(x) - 1 < d(y)$. Denote $G_0 = G[V(G) - \{x, y\}]$.

Construct a new graph $G^*$ such that $V(G^*) = V(G)$, and $E(G^*) = E(G) - \{v_t x\} + \{v_t y\}$. 
The function \( f(x) = \frac{x}{x + m} \) is strictly increasing for \( x \in [0, \infty) \), where \( m \) is a positive integer.

\[
\frac{\prod_{1,c}(G)}{\prod_{1,c}(G^*)} = \frac{[\prod_{w \in V(G_0)} d(w)^c] d(y)^c d(x)^c}{[\prod_{w \in V(G_0)} d(w)^c] [d(y) + 1]^c [d(x) - 1]^c}
\]

\[
= \frac{[d(y) + 1]^c [d(x) - 1]^c}{d(y)^c d(x)^c}
\]

\[
= \frac{[d(y) + 1]^c}{[d(x) - 1]^c}
\]

\[
> 1.
\]
The function \( g(x) = \frac{x^x}{(x + m)^{x+m}} \) is strictly decreasing for \( x \in [0, \infty) \), where \( m \) is a positive integer.

\[
\frac{\prod_2(G)}{\prod_2(G^*)} = \frac{\prod_{w \in V(G_0)} d(w)^{d(w)} d(y)^{d(y)} d(x)^{d(x)}}{\prod_{w \in V(G_0)} d(w)^{d(w)} [d(y) + 1]^{d(y)+1} [d(x) - 1]^{d(x)-1}}
\]
\[
= \frac{[d(y) + 1]^{d(y)+1} [d(x) - 1]^{d(x)-1}}{d(y)^{d(y)} d(x)^{d(x)}}
\]
\[
< 1.
\]
Proof of (ii)

If $d_{G'}(v) \geq k + 1$, reorder the subindices of $\{v_1, v_2...v_s\}$ so that $vv_i \notin E(G)$ with $i \in [1, s_1]$, where $s_1 \leq s$.

Construct a new graph $G^*$ such that $V(G^*) = V(G)$, and $E(G^*) = E(G) - \{uv_i\} + \{vv_i\}$, for all $i \in [1, s_1]$

Since $d(v) \geq k + s - s_1 + 1$ and $d(u) = k + s$, then $d(v) \geq d(u) - s_1 + 1$. 
Proof of (ii)

If \( d_{G'}(v) \geq k + 1 \), reorder the subindices of \( \{v_1, v_2...v_s\} \) so that \( vv_i \notin E(G) \) with \( i \in [1, s_1] \), where \( s_1 \leq s \).

Construct a new graph \( G^* \) such that \( V(G^*) = V(G) \), and \( E(G^*) = E(G) - \{uv_i\} + \{vv_i\} \), for all \( i \in [1, s_1] \).

Since \( d(v) \geq k + s - s_1 + 1 \) and \( d(u) = k + s \), then \( d(v) \geq d(u) - s_1 + 1 \).
\[
\frac{\prod_{1,c}(G)}{\prod_{1,c}(G^*)} = \frac{d(v)^c d(u)^c}{[d(v) + s_1]^c [d(u) - s_1]^c} = \frac{[d(v)^c]}{[d(v)+s_1]^c} = \frac{[d(u)-s_1]^c}{[d(u)]^c} > 1.
\]

\[
\frac{\prod_{2}(G)}{\prod_{2}(G^*)} = \frac{d(v)^{d(v)} d(u)^{d(u)}}{[d(v) + s_1]^{d(v)}+s_1 [d(u) - s_1]^{d(u)}-s_1} = \frac{[d(v)^{d(v)}]}{[d(v)+s_1]^{d(v)}+s_1} = \frac{[d(u)-s_1]^{d(u)}-s_1}{[d(u)]^{d(u)}} < 1.
\]
The proof of $\prod_{1,c}(T^k_n) \leq \prod_{1,c}(P^k_n)$ and $\prod_2(T^k_n) \geq \prod_2(P^k_n)$:
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- Let $G$ be a $k$-tree, assume that either $\prod_{1,c}(G)$ attains the maximum or $\prod_{2}(G)$ attains the minimum.
Sketch of the Proof

The proof of $\prod_{1,c}(T_n^k) \leq \prod_{1,c}(P_n^k)$ and $\prod_2(T_n^k) \geq \prod_2(P_n^k)$:

- Let $G$ be a $k$-tree, assume that either $\prod_{1,c}(G)$ attains the maximum or $\prod_2(G)$ attains the minimum.
- By contradiction, we can show that every hyper pendent edge is a $k$-path.
The proof of $\prod_{1,c}(T_k^n) \leq \prod_{1,c}(P_k^n)$ and $\prod_{2}(T_k^n) \geq \prod_{2}(P_k^n)$:

- Let $G$ be a $k$-tree, assume that either $\prod_{1,c}(G)$ attains the maximum or $\prod_{2}(G)$ attains the minimum.
- By contradiction, we can show that every hyper pendent edge is a $k$-path.
- By induction, we can prove that $|S_1(G)| = 2$, thus, $P_k^n$ attains the maximal $\prod_{1,c}(G)$ and minimal $\prod_{2}(G)$.
Thank you