Inductive tools for handling internally 4-connected binary matroids and graphs

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Matroids from graphs and matrices
The vertex-edge incidence matrix of $G$ (over the 2-element field):

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
a & 1 & 0 & 0 & 0 & 1 & 0 \\
b & 1 & 1 & 0 & 0 & 0 & 0 \\
c & 0 & 1 & 1 & 0 & 0 & 1 \\
d & 0 & 0 & 1 & 1 & 0 & 0 \\
e & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
$$
Matroids from graphs and matrices

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\begin{pmatrix}
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\end{pmatrix}
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\(E\) (the ground set): column labels

\(C\) (the circuits): minimal linearly dependent sets of columns
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We get a binary matroid \(M[A]\) from any matrix \(A\) over the 2-element field.
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We get a binary matroid \(M[A]\) from any matrix \(A\) over the 2-element field.

Every graphic matroid is binary.
A matroid is 2-connected if it does not break up as a 1-sum (direct sum).

Theorem (Tutte, 1966)

Let $M$ be a 2-connected matroid and $e$ be an element of $M$. Then $M \setminus e$ or $M / e$ is 2-connected.
A matroid is **2-connected** if it does not break up as a 1-sum (direct sum).
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Such matroids are also called **connected**.
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Let $G$ be a graph with at least three vertices and without isolated vertices.

$M(G)$ is 2-connected if and only if $G$ is 2-connected and loopless.
2-connected matroids and graphs

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A 2-connected matroid is 3-connected if it does not break up as a 2-sum.

Tutte's Wheels-and-Whirls Theorem (1966)

Let \( M \) be a 3-connected matroid. Then \( M \) has a 3-connected minor \( M' \) with \( |E(M) - E(M')| = 1 \) unless \( M \) is a wheel or a whirl. In the exceptional case, \( M \) has a 3-connected minor \( M' \) with \( |E(M) - E(M')| = 2 \).
3-connected graphs and matroids

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Let $G$ be a graph with at least four vertices and without isolated vertices.

$M(G)$ is 3-connected if and only if $G$ is 3-connected and simple.
3-connected graphs and matroids

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**Tutte's Wheels Theorem (1961)**

Let $G$ be a 3-connected simple graph. Then $G$ has a 3-connected simple minor $G'$ with

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unless $G$ is a wheel.

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4-connected matroids

A 3-connected binary matroid is internally 4-connected if it does not break up as a 3-sum. A 3-connected simple graph is internally 4-connected if it is 4-connected except for the possible presence of degree-3 vertices.
4-connected matroids

A matroid is 2-connected if it does not break up as a 1-sum.
4-connected matroids

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A matroid is 3-connected if it does not break up as a 2-sum.
4-connected matroids

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A matroid is 3-connected if it does not break up as a 2-sum.

**Question.** What is 3-sum for matroids?
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A 3-connected binary matroid is internally 4-connected if it does not break up as a 3-sum.

A 3-connected simple graph is internally 4-connected if it is 4-connected except for the possible presence of degree-3 vertices.
Let \( M \) be an internally 4-connected binary matroid. Then \( M \) has an internally 4-connected minor \( M' \) with \( 1 \leq |E(M) - E(M')| \leq 3 \) unless \( M \) or \( M^* \) is the cycle matroid of (i) a terrahawk; or (ii) a planar or Möbius quartic ladder.
Internally 4-connected binary matroids

A graph is internally 4-connected if and only if its cycle matroid is internally 4-connected.
Theorem (2011)

Let $M$ be an internally 4-connected binary matroid. Then $M$ has an internally 4-connected minor $M'$ with

$$1 \leq |E(M) - E(M')| \leq 3$$

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Internally 4-connected binary matroids

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(i) a terrahawk; or (ii) a planar or Möbius quartic ladder.
Corollary

Let $G$ be an internally 4-connected graph. Then $G$ has an internally 4-connected minor $G'$ with $1 \leq |E(G) - E(G')| \leq 3$ unless $G$ is:

(i) a terrahawk;
(ii) a planar or Möbius quartic ladder;
(iii) the planar dual of a planar quartic ladder.
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Seymour’s Splitter Theorem

Let $M$ be a 3-connected matroid and $N$ be a 3-connected proper minor of $M$. Then $M$ has a 3-connected minor $M'$ that has an $N$-minor such that $|E(M) - E(M')| = 1$ unless $M$ is a wheel or a whirl.

In the exceptional case, $M$ has a 3-connected minor $M'$ with an $N$-minor such that $|E(M) - E(M')| = 2$. 
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Seymour’s Splitter Theorem

Let $M$ be a 3-connected matroid and $N$ be a 3-connected proper minor of $M$.

**Goal.** To remove a small number of elements from $M$ and keep

(i) 3-connectedness; and

(ii) an isomorphic copy of $N$. 
Seymour’s Splitter Theorem (1980)

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A Splitter Theorem for internally 4-con. binary matroids

What is known for graphs?

Johnson and Thomas (2002)

Consequence. We cannot remove some bounded set of elements to recover internal 4-connectivity.
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Earlier work
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All allow the intermediate matroid to satisfy some weaker form of connectivity.
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Alternative approach. (Geelen) Expand the types of moves one allows to go from $M$ to $M'$. 
Another move

Delete all the dashed edges taking a \textit{quartic ladder segment} to a \textit{cubic ladder segment}.
A 3-separation in a matroid $M$ is a partition $(X, Y)$ such that $|X|, |Y| \geq 3$ and $r(X) + r(Y) - r(M) \leq 2$.

A 3-connected matroid is 4-connected if it has no 3-separations.

A 3-connected matroid is internally 4-connected if its only 3-separations $(X, Y)$ have $X$ or $Y$ equal to a triangle or a triad.

Other variants on 4-connectivity allow certain restricted kinds of 3-separations.
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Example. $(\{1, 2, 3, 4\}, \{5, 6, 7, 8\})$ is a 3-separation of the binary affine space $AG(3, 2)$. 

![Diagram of a binary affine space AG(3, 2)]
3-separations

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Other variants on 4-connectivity allow certain restricted kinds of 3-separations.
Towards a Splitter Theorem: Step 1

(i) $M$ and $N$ are internally 4-connected binary matroids; and
(ii) $N$ is a proper minor of $M$; and
(iii) $|E(N)| \geq 7$.

Theorem (2012) When $M$ is 4-connected, it has an internally 4-connected minor $M'$ that has an $N$-minor such that $|E(M) - E(M')| = 1$ (a 1-element WIN) unless $M$ is a certain 16-element self-dual matroid. In the exceptional case, there is a 2-element WIN.
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Suppose $M$ is 4-connected.

$M$ is an internally 4-connected matroid with no triangles and no triads.
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unless $M$ is a certain 16-element self-dual matroid. In the exceptional case, there is a 2-element WIN.
The 16-element exception

Let $D_{16}$ be the 16-element rank-8 matroid that is represented over $GF(2)$ by the matrix $[I_8|A]$ where $A$ is

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & | & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & | & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & | & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & | & 0 & 1 & 1 & 1 \\
\_ & \_ & \_ & \_ & | & \_ & \_ & \_ & \_ & \_ & \_ \\
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\]

Evidently $D_{16}$ is isomorphic to its dual. Moreover, $D_{16}$ has two visible $AG(3,2)$-minors on disjoint ground sets.
Step 2

We now get to assume that $M$ or $M^*$ has a triangle.

**Theorem**

Suppose every triangle and every triad of $M$ is retained in every $N$-minor. Then there is a 1- or 2-element win.

Now up to duality, $M$ has a triangle $T$ and an element $e$ in $T$ such that $N \preceq M \setminus e$.

Can we say something about the connectivity of $M \setminus e$?
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A 3-connected binary matroid is \((4, 4, S)\)-connected if one side of every 3-separation is a triangle, a triad, or a 4-element fan.
A weaker type of 4-connectivity

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Step 2 gave us a triangle containing an element \(e\) such that \(M \setminus e\) has an \(N\)-minor.
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We want to say something about the connectivity of \(M \setminus e\).
Theorem
Suppose $|E(M)| \geq 15$. Then
(i) there is a $1$-, $2$-, or $3$-element win; or
(ii) $M$ or $M^*$ is a cubic M"obius or planar ladder or a special single-element coextension thereof; or
(iii) up to duality, $M$ has a triangle $T$ and an element $e$ such that $M \setminus e$ has an $N$-minor and is $(4,4,S)$-connected.

What happens if (iii) of the last theorem holds?
Step 3

Recall our prevailing assumptions:

(i) $M$ and $N$ are internally 4-connected binary matroids; and
(ii) $N$ is a proper minor of $M$; and
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Theorem

Suppose $|E(M)| \geq 15$. Then (i) there is a 1-, 2-, or 3-element win; or (ii) $M$ or $M^*$ is a cubic Möbius or planar ladder or a special single-element coextension thereof; or (iii) up to duality, $M$ has a triangle $T$ and an element $e$ such that $M \setminus e$ has an $N$-minor and is $(4,4,S)$-connected.

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A $(4, 4, S)$-connected matroid has one side of every 3-separation as a triangle, a triad, or a 4-fan.
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What happens if (iii) of the last theorem holds?
Building structure around a good triangle

Assume $M$ has a triangle $T$ and an element $e$ such that $M \setminus e$ has an $N$-minor and is $(4, 4, S)$-connected.
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Figure: A good bowtie
Building structure around a good triangle

Assume $M$ has a triangle $T$ and an element $e$ such that $M \setminus e$ has an $N$-minor and is $(4, 4, S)$-connected.

Figure: An augmented 4-wheel: $M \setminus f$ has an $N$-minor.
Step 4

Theorem

Suppose $|E(M)| \geq 16$ and $M$ has a triangle $T$ and an element $e$ such that $M \setminus e$ has an $N$-minor and is $(4,4,S)$-connected. Then

(i) there is a 1-, 2-, 3-, or 4-element win; or

(ii) $M$ or $M^*$ has a good bowtie; or

(iii) $M$ or $M^*$ has an augmented 4-wheel.

Can we eliminate (iii)?
Theorem
Suppose $|E(M)| \geq 16$ and $M$ has a triangle $T$ and an element $e$ such that $M\setminus e$ has an $N$-minor and is $(4, 4, S)$-connected. Then

(i) there is a 1-, 2-, 3-, or 4-element win; or
(ii) $M$ or $M^*$ has a good bowtie; or
(iii) $M$ or $M^*$ has an augmented 4-wheel.

Can we eliminate (iii)?
Step 5: Killing augmented 4-wheels

Theorem

Suppose $|E(M)| \geq 16$ and $M$ has an augmented 4-wheel. Then

(i) there is a 1-, 2-, 3-, or 4-element win; or

(ii) $M$ has a good bowtie; or

(iii) $M$ is a terrahawk; or

(iv) $M$ contains the configuration shown below where the deletion of all of the green elements is an internally 4-connected matroid having an $N$-minor.
Step 5: Killing augmented 4-wheels

- $M \setminus e$ is $(4, 4, S)$-connected having an $N$-minor; and
- $M \setminus f$ has an $N$-minor.
Step 5: Killing augmented 4-wheels

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When bowties are good
Let $M$ have a good bowtie as shown, so $M \backslash e$ is $(4, 4, S)$-connected having an $N$-minor.
When bowties are good

Let $M$ have a good bowtie as shown, so $M\setminus e$ is $(4, 4, S)$-connected having an $N$-minor.

Then $M\setminus e$ has a 4-fan.
Let $M$ have a good bowtie as shown, so $M \setminus e$ is $(4, 4, S)$-connected having an $N$-minor.

Then $M \setminus e$ has a 4-fan.

As $N$ is internally 4-connected, it contains no 4-fans.
When bowties are good

As \( N \) is internally 4-connected, it contains no 4-fans.

**Lemma**

\( M \setminus e \setminus d \) or \( M \setminus e / a \) has an \( N \)-minor.
When bowties are good

Lemma
$M \setminus e \setminus d$ or $M \setminus e / a$ has an $N$-minor.

Lemma
If $M$ has a good bowtie, then it has a good bowtie in which $M \setminus e \setminus d$ has an $N$-minor.
Theorem

Suppose $M$ has a good bowtie in which $M - e$ has an $N$-minor.

Then (i) $M - d$ is $(4, 4, S)$-connected and $M$ contains one of the two structures shown; or (ii) $M - d$ is not $(4, 4, S)$-connected.

Case (ii) gives a WIN.
Theorem
Suppose $M$ has a good bowtie in which $M \setminus e \setminus d$ has an $N$-minor. Then

(i) $M \setminus d$ is $(4, 4, S)$-connected and $M$ contains one of the two structures shown; or

(ii) $M \setminus d$ is not $(4, 4, S)$-connected.
Building from a good bowtie

Theorem

Suppose $M$ has a good bowtie in which $M \setminus e \setminus d$ has an $N$-minor. Then

(i) $M \setminus d$ is $(4, 4, S)$-connected and $M$ contains one of the two structures shown; or

(ii) $M \setminus d$ is not $(4, 4, S)$-connected.

Case (ii) gives a WIN.
Building from a good bowtie

Theorem

When $M$ has a good bowtie in which $M$ has an $N$-minor but $M$ is not $(4, 4, S)$-connected, one of the following occurs:

(a) there is a $1$, $2$, $3$, or $4$-element win; or

(b) $M$ contains the configuration shown and deleting all of the green elements gives an internally $4$-connected matroid having an $N$-minor.
Building from a good bowtie

**Theorem**

When $M$ has a good bowtie in which $M \setminus e \setminus d$ has an $N$-minor but $M \setminus d$ is not $(4, 4, S)$-connected, one of the following occurs:

(a) there is a 1-, 2-, 3-, or 4-element win; or

(b) $M$ contains the configuration shown and deleting all of the green elements gives an internally 4-connected matroid having an $N$-minor.
Chains of bowties

We've seen chains of bowties.

We can also have rings of bowties.

Only expect small variants on these moves.
Chains of bowties

What other moves do we need?
Chains of bowties

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Bowtie moves:
Chains of bowties

**Bowtie moves:** We’ve seen chains of bowties.

![Diagram of chains of bowties]
Chains of bowties

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**Bowtie moves:** We’ve seen chains of bowties.

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Only expect small variants on these moves.
The endgame

What's left to do?

Handling chains of bowties.

Steps 6, 7, and 8

What do we get?

A theorem of the following form:

Given internally 4-connected binary matroids $M$ and $N$ such that $N$ is a proper minor of $M$, there is an internally 4-connected proper minor $M'$ of $M$ that has an $N$-minor such that $M'$ is obtained from $M$ by

(i) removing at most 4 elements; or

(ii) a bowtie or ladder move; or

(iii) a slight variant of a bowtie or ladder move.

Consequence: A new theorem for internally 4-connected graphs.
The endgame

What’s left to do?
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  (ii) a bowtie or ladder move; or

  (iii) a slight variant of a bowtie or ladder move.*
The endgame

What’s left to do? Handling chains of bowties.

Steps 6, 7, and 8

What do we get?

A theorem of the following form:

*Given* internally $4$-connected binary matroids $M$ and $N$ such that $N$ is a proper minor of $M$, there is an internally $4$-connected proper minor $M'$ of $M$ that has an $N$-minor such that $M'$ is obtained from $M$ by

(i) removing at most 4 elements; or

(ii) a bowtie or ladder move; or

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Consequence: A new theorem for internally $4$-connected graphs.
Summary

Two powerful inductive tools for 3-connected matroids and graphs.
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Tutte’s Wheels-and-Whirls Theorem
By removing at most two elements from a 3-connected matroid, we can recover 3-connectivity.

Seymour’s Splitter Theorem
By removing at most two elements from a 3-connected matroid, we can recover 3-connectivity and retain a copy of a 3-connected minor.

(1) By removing at most six elements from an internally 4-connected binary matroid, we can recover internal 4-connectivity.

(2) By removing elements from an internally 4-connected binary matroid using a small number of well-described moves, we can keep internal 4-conn. and a copy of an internally 4-conn. minor.
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Corresponding inductive tools for internally 4-connected binary matroids and graphs.
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