MA 4163 / 6163
Exam 1 – February 11, 2013

Name ________________________________

General Instructions: Please answer the following, showing all your work and writing neatly. You may not refer to any books, notes, or calculators.
Three problems, 75 total points.

1. (5 points each) Quickies
   (a) State the Isomorphism Theorem.
   (b) Let f be a "flip" element in D_8. List the right cosets of $\langle f \rangle$ in D_8, and identify a right transversal.
   (c) With notation as in Problem 1b, is $D_8 / \langle f \rangle$ a group under the set-wise product? Why or why not?
   (d) Find $|\mathrm{Aut}(\mathbb{Z}_{37})|$, the number of distinct automorphisms of $\mathbb{Z}_{37}$.
   (e) Recall that a *permutation group* on X is a nonempty subset of $\operatorname{Sym} X$ which is closed under composition and inverses. Show that a permutation group is a group.

2. Normal and characteristic subgroups
   (a) (13 points) Prove that if $C \operatorname{char} N$ and $N \operatorname{char} G$, then $C \operatorname{char} G$.
   (b) (12 points) Show that, for any subgroup H, we have $NC(H) \triangleq \langle H^g : g \in G \rangle$ (i.e., the subgroup generated by all conjugates of H) to be normal in G.

3. The Homomorphism Theorems
   (a) (13 points) If $N \subseteq H \subseteq G$, where $N \triangleleft G$ and H is a subgroup of G, then prove (without using the Correspondence Theorem) that $[G : H] = [G/N : H/N]$.
   (b) (12 points) Recall that $GL_2(\mathbb{F})$ is the set of 2×2 invertible matrices over the field $\mathbb{F}$, and that $SL_2(\mathbb{F})$ is the set of 2×2 matrices over $\mathbb{F}$ with determinant 1. Describe $GL_2(\mathbb{R})/SL_2(\mathbb{R})$ as isomorphic to a familiar group. (Prove that your answer works.)

      Hint: for any matrices A and B over any field $\mathbb{F}$, we have $\det(AB) = (\det A)(\det B)$.