MA 2733

Examination 2 – October 20, 2015

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5 T/F, several long answer. 50 points.

General Instructions: Please answer the following, without use of calculators. You may refer to a 3x5 card, but no other notes. Correct answers without correct supporting work may not receive full credit (excluding the True/False section). You may use the back of each page for additional answer space (please clearly indicate if you have done so), or scratch work.

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Signature ______

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

(a) T Since the series $\sum_{n=0}^{\infty} \frac{1}{n^4+2}$ converges, the integral $\int_0^{\infty} \frac{1}{x^4+2}\,dx$ also converges (by the Integral Test).

(b) F In the power series $\sum_{k=2}^{\infty} \frac{x^{2k}}{3k}$, the coefficient of x^2 is $1/3$.

(c) F If the power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 1, and $\lim_{n\to 0} c_n = 0$, then the interval of convergence is $[-1, 1]$.

(d) T If the alternating series $\sum_{n=0}^{\infty} (-1)^n b_n$ conditionally converges, then $\sum_{n=0}^{\infty} |b_{2n}| = \infty$.

(e) F The function $f(x) = \frac{1}{x}$ has a power series representation centered at $a = 0$.

2. Power series basics. Consider the power series $P(x) = \sum_{k=2}^{\infty} \frac{2^k}{(2k^2+1)} \cdot x^{2k+1}$.

(a) (2 points) Find the coefficients of x^0, x^3, x^6, and x^9 in $P(x)$.

$$P(x) = \frac{2^2}{2\cdot 4+1} \cdot x^5 + \frac{2^3}{2\cdot 9+1} x^7 + \frac{2^4}{2\cdot 16+1} x^9 + \ldots$$

$\Rightarrow$ coef of x^0 is 0

" " x^3 " 0

" " x^6 " 0

x^9 is $\frac{2^4}{2\cdot 16+1} = \frac{16}{33}$

(b) (4 points) Find the radius of convergence for $P(x)$.

RT: $$\lim_{k\to\infty} \left| \frac{\frac{2^{k+1}}{2(k+1)^2+1} \cdot x^{2k+3}}{\frac{2^k}{2k^2+1} x^{2k+1}} \right| = \lim_{k\to\infty} \frac{\frac{1}{k^2}(2k^2+1)}{\frac{1}{k^2}(2k^2+4k+3)} \cdot 2 \cdot |x^2|$$

$$= \lim_{k\to\infty} \frac{2+\frac{1}{k^2}}{2+\frac{4}{k}+\frac{3}{k^2}} \cdot 2 \cdot x^2 = 2x^2$$

and $2x^2 < 1 \Leftrightarrow |x| < \frac{1}{\sqrt{2}}$

So $R = \frac{1}{\sqrt{2}}$.

(c) (6 points) Find the interval of convergence for $P(x)$.

Plug in $x = \pm\frac{1}{\sqrt{2}}$:

$$P\left(\tfrac{1}{\sqrt{2}}\right) = \sum_{k=2}^{\infty} \frac{2^k}{2k^2+1} \cdot \frac{1}{(\sqrt{2})^{2k+1}}$$ and since $(\sqrt{2})^{2k} = 2^k$

$$= \sum_{k=2}^{\infty} \frac{\cancel{2^k}}{2k^2+1} \frac{1}{\cancel{2^k}\cdot\sqrt{2}}$$

and $\frac{1}{\sqrt{2}(2k^2+1)} < \frac{1}{k^2}$, so converges by DCT.

Similarly, $$P\left(-\tfrac{1}{\sqrt{2}}\right) = \sum_{k=2}^{\infty} \frac{2^k}{2k^2+1} \cdot \frac{1}{(-\sqrt{2})^{2k+1}}$$

$$= \sum_{k=2}^{\infty} \frac{-1}{\sqrt{2}(2k^2+1)} = -P\left(\tfrac{1}{\sqrt{2}}\right)$$ converges,

So interval of convergence is $\left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.

3. Discuss convergence of the following series: determine whether each is absolutely convergent, conditionally convergent, or divergent.

(a) (6 points) $\sum_{n=1}^{\infty} \frac{\sin n + \cos n}{3n + n^2}$

We have $\left| \frac{\sin n + \cos n}{3n + n^2} \right| \le \frac{2}{3n+n^2} \le \frac{2}{n^2}$ (and $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges)

So series converges absolutely by DCT.

(b) (5 points) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n} + \pi}$

We have $\sqrt{n} + \pi < 2\sqrt{n}$ for $n > \pi^2$

So $\frac{1}{\sqrt{n}+\pi} > \frac{1}{2\sqrt{n}}$.

As $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ diverges, series does not absolutely converge.

But $\frac{1}{\sqrt{n}+\pi}$ is +, decreasing with limit 0,

So series converges (conditionally) by AST.

(c) (5 points) $\sum_{n=3}^{\infty} \frac{(-1)^n}{n(\ln n)^3}$

The function $\frac{1}{x(\ln x)^3}$ is cts, +, decreasing, so IT applies to absolute convergence problem.

We evaluate

$$\int_3^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_{\ln 3}^{\infty} \frac{1}{u^3}\,du = \left[-\frac{u^{-2}}{2} \right]_{\ln 3}^{\infty}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \frac{(\ln 3)^{-2}}{2} - \frac{(\ln t)^{-2}}{2}$$

$$= \frac{(\ln 3)^{2}}{2}$$

and since $\int$ converges, series converges absolutely.

4. Power series representations.

(a) (4 points) Find a power series representation for $f(x) = \dfrac{1}{4+x^4}$.

(To receive full credit, isolate the powers of x in your series.)

$$\frac{1}{4+x^4} = \frac{1}{4}\,\frac{1}{1-\left(\frac{-x^4}{4}\right)} = \frac{1}{4}\sum_{k=0}^{\infty}\left(-\frac{x^4}{4}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{4^{k+1}}$$

(b) (2 points) On what interval does your representation from part (a) converge? Why?

When $\left|\frac{-x^4}{4}\right| < 1$, ie, when $x^4 < 4$

ie, on $(-\sqrt{2}, \sqrt{2})$.

5. (6 points) The "explain" problem

Explain why $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 \cos n}{4^n}$ converges, without using the Ratio Test.

Notice that, for $n \geq 4$, we have $\frac{n^2}{2^n} \leq 1$

(indeed, as $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$, $\frac{n^2}{2^n}$ is eventually smaller than any positive #).

Then, for $n \geq 4$, we have

$$\left|\frac{n^2 \cos n}{4^n}\right| \leq \frac{n^2}{4^n} = \frac{n^2}{2^n} \cdot \frac{1}{2^n} \leq \frac{1}{2^n}$$

and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a convergent geometric series,

so series converges absolutely by DCT.