MA 2733
Examination 3 – November 19, 2014

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5 T/F, several long answer. 50 points.

General Instructions: Please answer the following, without use of calculators. You may refer to a 3x5 card, but no other notes. Correct answers without correct supporting work may not receive full credit (excluding the True/False section). You may use the back of each page for additional answer space (please clearly indicate if you have done so), or scratch work.

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Signature ♡

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

   (a) F If $\vec{\mathbf{u}}$ is a vector, then $\dfrac{\vec{\mathbf{u}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}}$ is a unit vector.

   (b) F If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are vectors, then $\|\vec{\mathbf{u}} + \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\| + \|\vec{\mathbf{v}}\|$.

   (c) F If $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ are vectors, and $\vec{\mathbf{w}}$ is orthogonal to $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$, then either $\vec{\mathbf{w}} = \vec{\mathbf{v}} \times \vec{\mathbf{u}}$ or else $\vec{\mathbf{w}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

   (d) T If $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$ are orthogonal vectors, then $\|\vec{\mathbf{v}} \times \vec{\mathbf{u}}\| = \|\vec{\mathbf{v}}\| \cdot \|\vec{\mathbf{u}}\|$.

   (e) F If $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ are vectors, then $(\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}) \times \vec{\mathbf{w}}$ is orthogonal to $\vec{\mathbf{w}}$.

2. Lines and planes

(a) (4 points) Find a vector orthogonal to the plane $2x + 3y - 2z = 4$, and a point that lies on the plane.

$\vec{n} = \langle 2, 3, -2 \rangle$ (coefficients),

To find a point, set e.g. $x = y = 0$,

so equation becomes $-2z = 4$

$\Rightarrow (0, 0, -2)$ is on the plane.

(b) (4 points) Find 2 unit vectors that are parallel to the vector $\langle 3, -1, 4 \rangle$.

$\frac{\langle 3,-1,4 \rangle}{\sqrt{26}}$ and $-\frac{\langle 3,-1,4 \rangle}{\sqrt{26}}$

(c) (5 points) Do the points $(1, 2, 3)$, $(3, 1, 3)$, $(3, 2, 2)$, and $(5, 1, 2)$ lie on a common plane? If so, find the plane! If not, explain why not.

We find the plane containing the 1st 3 points:

2 vectors parallel are $\langle 2, -1, 0 \rangle = \langle 3,1,3 \rangle - \langle 1,2,3 \rangle$

and $\langle 2, 0, -1 \rangle = \langle 3,2,2 \rangle - \langle 1,2,3 \rangle$

cross product gives $\vec{n} = \langle 1, 2, 2 \rangle$

$\vec{n} \cdot (\vec{r} - \vec{r}_0)$ gives $\boxed{x + 2y + 2z = 11.}$

Since $(5,1,2)$ satisfies $5 + 2 \cdot 1 + 2 \cdot 2 = 11$,

all lie on this plane

(d) (4 points) Give the vector equation of a line that passes through the point $(0, 1, 1)$ and that is parallel to the line given by the symmetric equations

$$x - 1 = \frac{y-2}{2} = \frac{z-3}{3}.$$

Vector direction of the given line (from class)

is $\langle 1, 2, 3 \rangle$

Desired line is

$\vec{r}(t) = \langle 0, 1, 1 \rangle + t \langle 1, 2, 3 \rangle$

3. Vector functions and curves.

(a) (5 points) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ for the curve $\vec{\mathbf{r}}(t) = \langle t, \frac{t^2}{2}, \frac{t^3}{3} \rangle$.

$$\vec{r}\,'(t) = \langle 1, t, t^2 \rangle$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle 1, t, t^2 \rangle}{\sqrt{1+t^2+t^4}}$$

(b) (4 points) For the curve $\vec{\mathbf{r}}(t) = \langle 2\sin t, \cos t, -2t \rangle$, find $\vec{\mathbf{r}}'(t)$ and $\vec{\mathbf{r}}''(t)$.

$$\vec{r}\,'(t) = \langle 2\cos t, -\sin t, -2 \rangle$$

$$\vec{r}\,''(t) = \langle -2\sin t, -\cos t, 0 \rangle.$$

(c) (4 points) For the curve $\vec{\mathbf{r}}(t) = \langle 2\sin t, \cos t, -2t \rangle$ from part (b), find the plane which contains the point $\vec{\mathbf{r}}(\pi)$ and is parallel to the vectors $\vec{\mathbf{r}}'(\pi)$ and $\vec{\mathbf{r}}''(\pi)$.

$$\vec{r}(\pi) = \langle 0, -1, -2\pi \rangle$$

$$\vec{r}\,'(\pi) = \langle -2, 0, -2 \rangle$$

$$\vec{r}\,''(\pi) = \langle 0, 1, 0 \rangle$$

Then

$$\vec{n} = \vec{r}\,' \times \vec{r}\,'' = \langle 2, 0, -2 \rangle$$

so plane is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

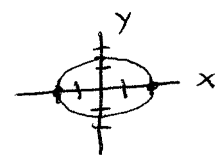
ie $2x - 2z = 4\pi$

(d) (4 points) Give a vector function representation for the curve given by the trace in the xy-plane of the curve $x^2 + 2y^2 + z^2 = 4$.

The xy-plane has equation $z=0$,

so trace is $x^2+2y^2=4$

which we recognize as equation of an ellipse. (pictured)



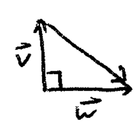
The vector equation of this ellipse we know from Chapter 10 to be

$\langle 2\cos t, \sqrt{2}\sin t\rangle$.

4. (6 points) The "explain" problem.
Using only the definition of dot product and elementary geometry, show that if the angle between 3-dimensional vectors $\vec{v}$ and $\vec{w}$ is $\pi/2$, then $\vec{v} \cdot \vec{w} = 0$.

Note: the definition of the dot product is the formula you usually use to compute it, and in particular does not involve any trig functions.

We make a triangle w/ $\vec{v}$ and $\vec{w}$:



where the hypotenuse is $\vec{w}-\vec{v}$

(since $\vec{v} + \underbrace{(\vec{w}-\vec{v})}_{\text{hyp}} - \vec{w} = 0$).

By Pythagorean Theorem,

$$\|\vec{w}-\vec{v}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2.$$

We calculate also w/ dot product:

$$\begin{aligned}\|\vec{w}-\vec{v}\|^2 &= (\vec{w}-\vec{v})\circ(\vec{w}-\vec{v}) \\ &= \vec{w}\circ\vec{w} - 2\vec{w}\circ\vec{v} + \vec{v}\circ\vec{v} \\ &= \|\vec{w}\|^2 - 2\vec{w}\circ\vec{v} + \|\vec{v}\|^2.\end{aligned}$$

But the 2 expressions are equal to one another!

So $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{w}\|^2 - 2\vec{w}\circ\vec{v} + \|\vec{v}\|^2$.

Cancelling and solving for $\vec{w}\circ\vec{v}$,

we see $\vec{w}\circ\vec{v} = 0$.