MA 2733
Examination 2 – October 22, 2014

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5 T/F, several long answer. 50 points.

**General Instructions:** Please answer the following, without use of calculators. You may refer to a 3x5 card, but no other notes. Correct answers without correct supporting work may not receive full credit (excluding the True/False section). You may use the back of each page for additional answer space (please clearly indicate if you have done so), or scratch work.

**Mississippi State University Honor Code:** "As a Mississippi State University student I will conduct myself with honor and integrity at all times. I will not lie, cheat, or steal, nor will I accept the actions of those who do."

Signature HA

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

(a) T If $\frac{1}{n^3} \le a_n \le \frac{1}{n^{3/2}}$ for all $n \ge 3$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) F If $\frac{1}{n^2} \le a_n \le \frac{1}{\sqrt{n}}$ for all $n \ge 3$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(c) T Every function that can be differentiated infinitely many times has a Taylor series.

(d) F The coefficient of $x^{10}$ in $\sum_{m=1}^{\infty} m \cdot x^{2m}$ is 10.

(e) T If the power series $\sum_{n=0}^{\infty} c_n x^n$ converges at $x = 4$, then it must converge at $x = -\pi$.

2. (9 points) Discuss convergence of the series $\sum_{n=0}^{\infty} \frac{-\cos n}{2^n+1}$. (Determine whether it is absolutely convergent, conditionally convergent, or divergent.)

Since $\left|\frac{-\cos n}{2^n+1}\right| \le \frac{|\cos n|}{2^n+1} \le \frac{1}{2^n+1} \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$,

the series converges absolutely by Direct Comparison w/ the geometric series $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

3. Power series convergence: $\sum_{n=0}^{\infty} \frac{e^{-n}}{\sqrt{n}+2} \cdot x^n$

(a) (6 points) Find the radius of convergence of the above power series.

We apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{e^{-(n+1)}}{\sqrt{n+1}+2} \cdot x^{n+1}}{\frac{e^{-n}}{\sqrt{n}+2} \cdot x^n} \right| = \lim_{n\to\infty} \frac{e^{-n-1}}{e^{-n}} \cdot \frac{\sqrt{n}+2}{\sqrt{n+1}+2} \cdot \left|\frac{x^{n+1}}{x^n}\right|$$

$$= \lim_{n\to\infty} \frac{|x|}{e} \cdot \frac{\sqrt{n}+2}{\sqrt{n+1}+2} = \lim_{n\to\infty} \frac{|x|}{e} \cdot \frac{\underbrace{\frac{\sqrt{n}}{\sqrt{n}}}_{1} + \underbrace{\frac{2}{\sqrt{n}}}_{0}}{\underbrace{\sqrt{\frac{n+1}{n}}}_{1} + \underbrace{\frac{2}{\sqrt{n}}}_{0}} = \frac{|x|}{e}$$

Compare: $\frac{|x|}{e} < 1 \iff |x| < e$

So Radius is $\boxed{e}$.

(b) (6 points) Find the interval of convergence of the above power series.

Check endpts:

$x=-e$ : $\sum_{n=0}^{\infty} \frac{e^{-n}}{\sqrt{n+2}} \cdot (-e)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$.

Since $\frac{1}{\sqrt{n+2}}$ is positive decreasing limit 0, series converges by AST, ✓

$x=e$: $\sum_{n=0}^{\infty} \frac{e^{-n}}{\sqrt{n}+2} \cdot e^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}+2}$,

and $\sqrt{n}+2 \leq 3\sqrt{n}$ for $n \geq 1$

$\Rightarrow \frac{1}{\sqrt{n}+2} \geq \frac{1}{3\sqrt{n}}$,

and $\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}}$ diverges, hence $\sum_{n=0} \frac{1}{\sqrt{n}+2}$ does also, (Direct Comparison) ✓

4. (6 points) The "explain" problem. Show that if $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Absolutely convergent means that $\sum_{n=0} |a_n|$ converges

Then $-|a_n| \leq a_n \leq |a_n|$,

so $0 \leq a_n + |a_n| \leq 2|a_n|$

and Direct Comparison gives that $\sum_{n=0} a_n + |a_n|$ converges,

Then $\sum_{n=0}^{\infty} a_n = \left(\sum_{n=0}^{\infty} a_n + |a_n|\right) - \left(\sum_{n=0}^{\infty} |a_n|\right)$

is difference of 2 convergent series, hence converges. ▣

5. Power series applications

(a) (5 points) Find a power series representation for $f(x) = e^{3x^2}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow e^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} \cdot x^{2n}$$

(b) (5 points) Using your answer from part (a), find a series representation for $\int_0^1 e^{3x^2}\, dx$.

$$\int e^{3x^2} dx = C + \sum_{n=0}^{\infty} \frac{3^n}{n!} \cdot \frac{x^{2n+1}}{2n+1}$$

$$\Rightarrow \int_0^1 e^{3x^2} dx = \sum_{n=0}^{\infty} \frac{3^n}{n!} \cdot \frac{1}{2n+1}$$

(Plugging in to Evaluation Theorem).

(c) (3 points) Using your answer from part (a) and Taylor's Theorem, find $f^{(99)}(0)$ and $f^{(100)}(0)$ for $f(x) = e^{3x^2}$.

$$e^{3x^2} = \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}_{\text{Taylor's Thm}} = \underbrace{\sum_{k=0}^{\infty} \frac{3^k}{k!} x^{2k}}_{\text{part (a)}}$$

$$\Rightarrow \frac{f^{(99)}(0)}{99!} = \text{coef of } x^{99} = 0 \Rightarrow f^{(99)}(0) = 0$$

and $\frac{f^{(100)}(0)}{100!} = \text{coef of } x^{100} = \frac{3^{50}}{50!} \Rightarrow f^{(100)}(0) = \frac{3^{50}}{50!} \cdot 100!$