MA 2733
Examination 1 – September 24, 2014

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5 T/F, several long answer. 50 points.

General Instructions: Please answer the following, without use of calculators. You may refer to a 3x5 card, but no other notes. Correct answers without correct supporting work may not receive full credit (excluding the True/False section). You may use the back of each page for additional answer space (please clearly indicate if you have done so), or scratch work.

Mississippi State University Honor Code: "As a Mississippi State University student I will conduct myself with honor and integrity at all times. I will not lie, cheat, or steal, nor will I accept the actions of those who do."

Signature

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

   (a) T $\sum_{n=1}^{\infty} a_n = L$ means that $\lim_{m\to\infty} \sum_{n=1}^{m} a_n = L$.

   (b) T Assuming that the series $\sum_{n=1}^{\infty} \frac{1}{n^4 - \pi}$ converges, the integral $\int_2^{\infty} \frac{1}{x^4 - \pi}\, dx$ must also converge by the Integral Test.

   (c) T As $n \to \infty$, the sequence $\frac{1}{\sqrt{n}}$ converges.

   (d) F The series' $\sum_{n=0}^{\infty} \frac{a_n}{2n+1}$ and $\sum_{n=1}^{\infty} \frac{a_{n-1}}{2n+1}$ are the same.

   (e) T If a_n is an increasing sequence that converges to 12, then a_n is bounded.

2. Parametric and polar equations

(a) (7 points) Find the tangent line to $r = 2\sin\theta$ at $\theta = \frac{\pi}{3}$.

First calculate $\dfrac{dy}{dx} = \dfrac{\frac{d}{d\theta}[2\sin\theta \cdot \sin\theta]}{\frac{d}{d\theta}[2\sin\theta \cdot \cos\theta]} = \dfrac{4\sin\theta\cos\theta}{2\cos^2\theta - 2\sin^2\theta}$.

Plug in $\theta = \frac{\pi}{3}$ to find slope:

$$\text{slope} = \frac{4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}}{2 \cdot \frac{1}{4} - 2 \cdot \frac{3}{4}} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

Plug into point-slope formula: $(x = 2\sin\theta\cos\theta,\ y = 2\sin^2\theta)$

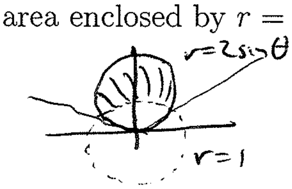
$$y - \frac{3}{2} = -\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right).$$

✓

(b) (7 points) Find the area that is inside the polar curve $r = 2\sin\theta$, but outside the polar curve $r = 1$.

At least 3 points will be given for finding the area enclosed by $r = 2\sin\theta$.

Sketch (not required for credit):



Solve for where curves $r = 2\sin\theta$ and $r = 1$ meet:

Set $2\sin\theta = 1$

$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \sin\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

Area is

$$\int_{\pi/6}^{5\pi/6} \frac{1}{2}(2\sin\theta)^2\, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2} \cdot 1^2\, d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 1 - \cos 2\theta\, d\theta - \frac{\pi}{3}$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/6}^{5\pi/6} - \pi/3$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{6}\right) - \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) - \pi/3 = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

optional simplification

✓

3. The "explain" problem.

(a) (2 points) State the arc-length formula for a parametric curve $x = f(t)$, $y = g(t)$. Make sure to explain any additional unknowns ('letters' beyond x, y, t, f, g) that you use.

$$L = \text{arc length} = \int_{\alpha}^{\beta} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

where $[\alpha, \beta]$ is the interval of t-values on which we are examining the curve.

(b) (5 points) Starting from the formula in (a), find the arc-length for a polar curve $r = h(\theta)$. (Explain why your formula works!)

Translate to Cartesian: $x = h(\theta)\cos\theta$, $y = h(\theta)\sin\theta$

$$\Rightarrow \frac{dx}{d\theta} = h'(\theta)\cos\theta - h(\theta)\sin\theta, \quad \frac{dy}{d\theta} = h'(\theta)\sin\theta + h'(\theta)\cos\theta$$

Plug into (a):

$$L = \int_c^d \sqrt{\begin{array}{l} h'(\theta)^2\cos^2\theta - 2h'(\theta)h(\theta)\cos\theta\sin\theta + h(\theta)^2\sin^2\theta \\ + h'(\theta)^2\sin^2\theta + 2h'(\theta)h(\theta)\sin\theta\cos\theta + h(\theta)^2\cos^2\theta \end{array}}\, d\theta$$

Cancel the middle terms, apply Pythagorean Thm $(\sin^2\theta + \cos^2\theta = 1)$ on outer terms to get

$$L = \int_c^d \sqrt{(h'(\theta))^2 + (h(\theta))^2}\, d\theta.$$

4. Discuss convergence of the following series: determine whether each is convergent or divergent.

(a) (6 points) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{\sqrt{n^2+1}}$.

nth Term Test: $\displaystyle\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}}} = 1 \neq 0$

So series diverges.

Partial fractions calculation:

$$\frac{1}{n^2-1} = \frac{1}{(n-1)(n+1)} = \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$$

(b) (7 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

Solution 1: Telescoping: Series $= \displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{2(n-1)} - \frac{1}{2(n+1)}\right)$ by above calculation

So Nth Partial Sum is

$$S_N = \left(\frac{1}{2} - \cancel{\frac{1}{6}}\right) + \left(\frac{1}{4} - \cancel{\frac{1}{8}}\right) + \left(\cancel{\frac{1}{6}} - \frac{1}{10}\right) + \left(\cancel{\frac{1}{8}} - \frac{1}{12}\right) + \dots$$

$$+ \left(\frac{1}{2(N-2)} - \frac{1}{2N}\right) + \left(\frac{1}{2(N-1)} - \frac{1}{2(N+1)}\right)$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2N} - \frac{1}{2(N+1)}$$

and $\lim_{N\to\infty} S_N = \frac{1}{2} + \frac{1}{4} - 0 - 0 = \frac{3}{4}$, so series converges (to $\frac{3}{4}$). ✓

Solution 2: Integral Test. $\frac{1}{x^2-1}$ is clearly cts, positive, decreasing, on $[2,\infty)$

By Integral Test, equiconvergent w/ $\int_2^\infty \frac{1}{x^2-1}\,dx$

By Partial Fractions, integral $= \int_2^\infty \frac{1}{2x-2} - \frac{1}{2x+2}\,dx$

$$= \left[\frac{\ln(2x-2)}{2} - \frac{\ln(2x+2)}{2}\right]_2^\infty$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(\frac{2x-2}{2x+2}\right)\right]_2^b$$

$$= \lim_{b\to\infty} \frac{1}{2}\ln\left(\frac{2b-2}{2b+2}\right) - \frac{1}{2}\ln\frac{1}{2}$$

$\frac{2b-2}{2b+2} \to 1$

$$= 0 - \frac{1}{2}\ln\frac{1}{2} = \frac{1}{2}\ln 2$$

So Integral converges, + series does also. ✓

(Solution 3 by Direct Comparison was discussed in class.)

(c) (6 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$.

Integral Test:

x, $\ln x$ and $x^{3/2}$ are all increasing on $[2,\infty)$

So $\frac{1}{x(\ln x)^{3/2}}$ is decreasing on $[2,\infty)$ (clearly positive and cts),

$\overset{\text{Integral Test}}{\Longrightarrow}$ Series is equiconv w/ $\int_2^\infty \frac{1}{x(\ln x)^{3/2}}$. Substitute $u = \ln x$, $du = \frac{1}{x}dx$

$$= \int_{\ln 2}^{\infty} \frac{1}{u^{3/2}}\,dx$$

$$= \left[\frac{u^{-1/2}}{-1/2}\right]_{\ln 2}^{\infty}$$

$$= \lim_{b\to\infty}\left[\frac{-2}{\sqrt{u}}\right]_{\ln 2}^{b}$$

$$= \lim_{b\to\infty} \frac{-2}{\sqrt{b}} + \frac{2}{\sqrt{\ln 2}} = \frac{2}{\sqrt{\ln 2}}.$$

So Integral + Series both converge.