MA 2733

Examination 3 – November 20, 2013

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5 T/F, several long answer. 50 points.

**General Instructions:** Please answer the following, without use of calculators. You may refer to a 3x5 card, but no other notes. Correct answers without correct supporting work may not receive full credit (excluding the True/False section). You may use the back of each page for additional answer space (please clearly indicate if you have done so), or scratch work.

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Signature

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

   ~~(a) ____ If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} |a_{n+1}/a_n| \leq 1$.~~

   False, since limit may not exist, but True if limit exists. Discounted from grade

   (b) T Suppose $a_n \leq 0$ for all $n$. If the series $\sum_{n=0}^{\infty} a_n$ converges, then the series converges absolutely. Since $\sum_{n=0}^{\infty} |a_n| = -\sum_{n=0}^{\infty} a_n$.

   (c) T We can consider $f(x) = x^{10}$ to be a power series.

   (d) T In the power series $f(x) = \sum_{k=0}^{\infty} k \cdot x^k$, the coefficient of $x^5$ is 5.

   (e) F If $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \leq \frac{1}{n^2}$ for all $n \geq A$ (for some $A$).

   Ex: $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

2. Discuss convergence of the following series: determine whether each is absolutely convergent, conditionally convergent, or divergent.

(a) (6 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$.

Since $0 < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n-1}}$ for $n \geq 2$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty$

the series diverges by DCT.

(b) (6 points) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+3n-5}{2^n}$.

Apply Ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2+3(n+1)-5}{2^{n+1}}}{\frac{n^2+3n-5}{2^n}} \right| = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \cdot \frac{n^2+5n-1}{n^2+3n-5}$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{1+\frac{5}{n}-\frac{1}{n^2}}{1+\frac{3}{n}-\frac{5}{n^2}} = \frac{1}{2}\cdot 1 = \frac{1}{2} < 1$$

So series converges (absolutely)

(c) (6 points) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cos n}{2^n}$.

Since $\left| \frac{(-1)^n \cos n}{2^n} \right| = \frac{|\cos n|}{2^n} \leq \frac{1}{2^n}$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (as a geometric series w/ $r = \frac{1}{2}$ )

the series converges absolutely.

(d) (8 points) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$.

First: $\sqrt{n+n} > \sqrt{n+1}$ for $n \geq 1$

So $\displaystyle \frac{1}{\sqrt{2n}} < \frac{1}{\sqrt{n+1}} = \left|\frac{(-1)^n}{\sqrt{n+1}}\right|$ for $n \geq 1$.

and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}} = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges,

by DCT the series does not absolutely converge

Then: apply AST.

$\displaystyle \frac{1}{\sqrt{n+1}} > 0$ (alternating)

$\displaystyle \frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{n+1+1}}$ (decreasing)

and $\displaystyle \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$

So the series converges conditionally

3. (6 points) On what interval does the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$ converge?

(0 points for correct interval, 6 points for showing the power series converges on the interval, and diverges off of it.)

There are a variety of approaches to this problem.

Perhaps the easiest:

Apply Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)2^{n+1}}}{\frac{x^n}{n \cdot 2^n}} \right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n}{n+1}$$

$$= |x| \cdot \frac{1}{2} \cdot 1 = \frac{|x|}{2}$$

and $\displaystyle \frac{|x|}{2} < 1 \iff |x| < 2$

So series converges absolutely for $|x| < 2$,

and diverges for $|x| > 2$ (by similar argument)

It remains to check $x = \pm 2$.

At $x = 2$ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$x = -2$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n\, 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally

(both from class)

So series converges exactly on interval $[-2, 2)$.

4. The "explain" problem.

(a) (4 points) Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution1: Integral test. $\frac{1}{x}$ is cts, +, decreasing on $[1,\infty)$

so series is equiconvergent w/ $\int_1^\infty \frac{1}{x}\,dx$

$= [\ln x]_1^\infty = \lim_{t\to\infty} \ln t - \ln 1 = \infty$

and since integral diverges, the series does also.

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Solution2: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \underbrace{\frac{1}{2}}_{\geq 1/2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 1/2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 1/2} + \cdots$

and continuing like this (considering blocks of $2^k$ terms) we see that

$\sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{n=1}^{\infty} \frac{1}{2} = \infty$, as desired.

(b) (4 points) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Solution1: Integral Test. $\frac{1}{x^2}$ is cts, +, decreasing on $[1,\infty)$

So series is equiconvergent w/ $\int_1^\infty \frac{1}{x^2}\,dx$

$= [-x^{-1}]_1^\infty = \lim_{t\to\infty} \frac{-1}{t} + 1 = 0 + 1 = 1$

and since integral converges, the series does also.

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Solution2: $0 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{2}{n^2+n}$ (since $n^2+n < 2n^2 \Rightarrow \frac{n^2+n}{2} < n^2$ )

and $\sum_{n=1}^{\infty} \frac{2}{n^2+n} = \sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{2}{n+1}\right)$

$= \lim_{N\to\infty} 2 - \frac{2}{N+1}$ (Telescopes)

$= 2$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$, it converges.