MA 2733

Examination 2 – October 23, 2013

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5 T/F, several long answer. 50 points.

General Instructions: Please answer the following, without use of calculators. You may refer to a 3x5 card, but no other notes. Correct answers without correct supporting work may not receive full credit (excluding the True/False section). You may use the back of each page for additional answer space (please clearly indicate if you have done so), or scratch work.

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Signature

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

   (a) T If (a, b, c) is a point on the plane $\alpha x + \beta y + \gamma z = 0$, then $\langle a, b, c\rangle$ is orthogonal to $\langle \alpha, \beta, \gamma\rangle$.

   (b) F The twisted cubic $\vec{\mathbf{r}}(t) = \langle t, t^2, t^3\rangle$ does not have a parametrization according to arc length.

   (c) T If $\vec{\mathbf{r}}(t)$ is any vector function, then $\|\vec{\mathbf{r}}\| = \sqrt{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}}$.

   (d) F Suppose f is a continuous function with $\lim_{x\to n} f(x) = 0$. If $a_n = f(n)$, then $\lim_{n\to\infty} a_n = 0$.

   (e) F If a_n is a bounded sequence, then $\lim_{n\to\infty} a_n$ converges.

2. (10 points) Find the curvature of the vector function $\vec{\mathbf{r}}(t) = \left\langle t, \frac{t^2}{2}, \frac{t^3}{3} \right\rangle$ (as a function of t).

We calculate:

$$\vec{r}\,'(t) = \langle 1, t, t^2 \rangle$$
$$\vec{r}\,''(t) = \langle 0, 1, 2t \rangle$$
$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \langle 2t^2 - t^2, 0 - 2t, 1 - 0 \rangle$$
$$= \langle t^2, -2t, 1 \rangle$$

So

$$K(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{\sqrt{t^4 + 4t^2 + 1}}{(1 + t^2 + t^4)^{3/2}}$$

3. Sequences

(a) (3 points) Find $\lim_{n\to\infty} \frac{1}{n}$. (Make sure to briefly explain your answer!)

$\lim_{n\to\infty} \frac{1}{n} = 0$, since numbers in denominator grow arbitrarily large (to ∞).

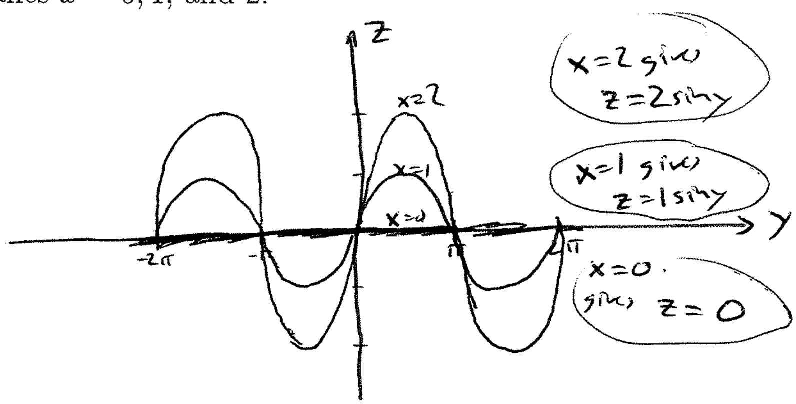
(b) (8 points) Find an explicit formula for the sequence a_n which is given recursively by $a_n = 2n \cdot a_{n-1}$ for $n > 1$, with $a_1 = 5$.

$$\begin{aligned} a_n = 2n \cdot a_{n-1} &= 2n \cdot 2(n-1) \cdot a_{n-2} \\ &= 2^2 \cdot n(n-1)\, a_{n-2} \\ &= 2^3\, n(n-1)(n-2)\, a_{n-3} \\ &= \cdots \\ &= 2^{n-1}\, n \cdot (n-1) \cdots 3 \cdot 2 \cdot a_1 \\ &= 2^{n-1}\, n! \cdot 5 \end{aligned}$$

✓

4. Planes and surfaces

(a) (7 points) On the same axis, draw the traces of the surface $z = x \sin y$ in the planes $x = 0, 1,$ and 2.



x=2 sin() $z = 2\sin y$

x=1 sin() $z = 1\sin y$

x=0. sin() $z = 0$

(b) (5 points) Find 2 unit vectors parallel to the plane $2x - y - z = 2$.
Partial credit will be given for finding several distinct points on the plane.

Suffices to find 2 vectors orthogonal to the normal vector $\langle 2,-1,-1\rangle$, then normalize to be unit vectors.

Eg. $\langle 2,-1,-1\rangle \cdot \langle 1,2,0\rangle = 0$

so $\dfrac{\langle 1,2,0\rangle}{\sqrt{5}}$ is as desired,

$\langle 2,-1,-1\rangle \cdot \langle 1,0,2\rangle = 0$

so $\dfrac{\langle 1,0,2\rangle}{\sqrt{5}}$ is also as desired.

5. (7 points) The "explain" problem.
Let $\vec{\mathbf{r}}(t)$ be a smooth vector function such that $\vec{\mathbf{r}}'$ is also smooth. Explain why

$$\vec{\mathbf{r}}''(t) = \nu'\vec{\mathbf{T}} + \nu^2\kappa\vec{\mathbf{N}}.$$

At least 1 point will be given for stating clearly what ν and κ represent in this equation.

By definition, $\vec{T} = \dfrac{\vec{r}\,'}{\|\vec{r}\,'\|}$, where we write $\nu = \|\vec{r}\,'\|$, so

equivalently $\vec{T} = \dfrac{\vec{r}\,'}{\nu}$

$\Rightarrow \vec{r}\,' = \nu\vec{T}$

Differentiating, we get

$\vec{r}\,'' = \nu'\vec{T} + \nu\vec{T}\,'$

but $\vec{N} = \dfrac{\vec{T}\,'}{\|\vec{T}\,'\|} \Rightarrow \vec{T}\,' = \|\vec{T}\,'\|\,\vec{N}$

Also, $K = \dfrac{\|\vec{T}\,'\|}{\|\vec{r}\,'\|} = \dfrac{\|\vec{T}\,'\|}{\nu} \Rightarrow \|\vec{T}\,'\| = K\nu$

Thus,

$\vec{r}\,'' = \nu'\vec{T} + \nu\|\vec{T}\,'\|\,\vec{N} = \nu'\vec{T} + \nu K \nu \vec{N}$

$= \nu'\vec{T} + \nu^2 K \vec{N}$

as desired.