MA 2733

Examination 1 – September 25, 2013

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5 T/F, 3 long answer. 50 points.

General Instructions: Please answer the following, without use of calculators. You may refer to a 3x5 card, but no other notes. Correct answers without correct supporting work may not receive full credit (excluding the True/False section). You may use the back of each page for additional answer space (please clearly indicate if you have done so), or scratch work.

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Signature ______

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

   (a) F If $\vec{v}$ and $\vec{u}$ are orthogonal vectors, then $\|\vec{v} \times \vec{u}\| = 1$.

   e.g. $\langle 2,0,0\rangle \times \langle 0,2,0\rangle$

   (b) T The curve described by the symmetric equation

   $$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

   can also be described with a parametric equation.

   As can any straight line.

   (c) F The Right-Hand Rule shows that $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$.

   RH rule determines direction of $\vec{v} \times \vec{w}$.

   (d) F If $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$, then $\vec{c}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

   rather orthogonal to $\vec{a} \times \vec{b}$

   (e) T If $\vec{a} \cdot \vec{a} = 1$, then $\vec{a}$ is a unit vector.

   Since $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

2. Parametric and polar equations

(a) (4 points) Set up an integral for the arc length of the polar curve given by $r = 2\theta \sin\theta$ for θ on the interval $[0, \pi/2]$. (You need not evaluate the integral.)

$$\frac{dr}{d\theta} = 2\sin\theta + 2\theta\cos\theta \qquad \text{(Product Rule)}$$

So $$L = \int_0^{\pi/2} \sqrt{4\theta^2\sin^2\theta + (2\sin\theta + 2\theta\cos\theta)^2}\, d\theta$$

(b) (7 points) Find the area inside the polar curve $r = \sqrt{\theta}\sin\theta$ for θ between 0 and π.

$$A = \int_0^{\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_0^{\pi} \theta \sin\theta\, d\theta = \frac{1}{2}\Big[-\theta\cos\theta\Big]_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos\theta\, d\theta$$

$$\boxed{\begin{array}{ll} \multicolumn{2}{l}{\text{Integration by parts:}} \\ u = \theta & dv = \sin\theta\, d\theta \\ du = d\theta & v = -\cos\theta \end{array}}$$

$$= \frac{1}{2}\Big[-\theta\cos\theta + \sin\theta\Big]_0^{\pi} = \frac{1}{2}\big((+\pi - 0) - (-0 + 0)\big) = \frac{\pi}{2}$$

(c) (3 points) In 1-2 sentences, explain under what conditions it is possible to write a parametric curve as a function $y = f(x)$.

When the parametric curve passes the vertical line test.

(d) (7 points) By calculating an appropriate 2nd derivative, show that the cycloid described by $x = 3t - 3\sin t$, $y = 3 - 3\cos t$ is concave down except for at the points where t is a multiple of 2π.
At least 3 points will be given for correctly calculating an appropriate 1st derivative.

First:
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\sin t}{3-3\cos t} = \frac{\sin t}{1-\cos t}$$
$$\left(\frac{dx}{dt} = 3(1-\cos t)\right)$$

Then:
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(dy/dx)/dt}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)}{3(1-\cos t)}$$

$$\overset{\text{Quotient Rule}}{=} \frac{\frac{\cos t(1-\cos t)-\sin^2 t}{(1-\cos t)^2}}{3(1-\cos t)} = \frac{\cos t-\cos^2 t-\sin^2 t}{3(1-\cos t)^3}$$

$$= \frac{\cos t-1}{3(1-\cos t)^3} = \frac{-1}{3(1-\cos t)^2} \qquad (t \neq \text{a multiple of } 2\pi)$$

The cycloid function is concave down exactly where its 2nd derivative is negative. The above-calculated 2nd derivative is clearly negative everywhere it is defined. It fails to be defined when $1-\cos t = 0$ ie, when $\cos t = 1$, ie when t is a multiple of 2π.

3. Vectors and straight-line geometry

(a) (5 points) Give the vector equation of a line which passes through the points $(1, 2, 3)$ and $(7, -3, 4)$.

$\langle 7,-3,4\rangle - \langle 1,2,3\rangle = \langle 6,-5,1\rangle$

So
$$\boxed{\vec{r}(t) = t\langle 6,-5,1\rangle + \langle 1,2,3\rangle}$$

(b) (4 points) Find the normal vector to and a point on the plane given by equation $x - y - z = 1$.

Normal vector $\vec{n} = \langle 1, -1, -1 \rangle$ (from coefficients of x, y, z)

The point $(1, 0, 0)$ satisfies the above equation (so is on the plane).

(c) (4 points) Find a unit vector $\vec{\mathbf{v}}$ which is orthogonal to $\langle 3, 1 \rangle$, and a unit vector $\vec{\mathbf{w}}$ which is parallel to $\langle 3, 1 \rangle$.

$$\vec{w} = \frac{\langle 3, 1 \rangle}{\|\langle 3, 1 \rangle\|} = \frac{\langle 3, 1 \rangle}{\sqrt{10}}$$

$\vec{u}$ is orthogonal to $\langle 3, 1 \rangle$ $\iff$ $3u_1 + 1u_2 = 0$

$\iff$ $u_2 = -3u_1$

So $\langle 1, -3 \rangle$ is orthogonal to $\langle 3, 1 \rangle$

$$\Rightarrow \vec{v} = \frac{\langle 1, -3 \rangle}{\|\langle 1, -3 \rangle\|} = \frac{\langle 1, -3 \rangle}{\sqrt{10}}$$

4. (6 points) The "explain why"/proof problem
Explain why $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is orthogonal to $\vec{\mathbf{a}}$. (You may use the lemma from class if you find it helpful.)

By the lemma, $(\vec{a} \times \vec{b}) \cdot \vec{a} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

and we calculate this determinant to be 0.

A 0 dot product "detects" orthogonality of $\vec{a} \times \vec{b}$ and $\vec{a}$.