MA 2733
Examination 3 Solutions – December 1, 2012

5 T/F, 4 long answer. 50 points.

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

   (a) F If $\lim_{n\to\infty} b_n = 0$, then the series $\sum_{n=1}^{\infty} b_n$ converges.

   Counterexample: $\sum \frac{1}{n}$.

   (b) F If the series $\sum_{n=1}^{\infty} b_n$ converges, then the series $\sum_{n=1}^{\infty} |b_n|$ converges.

   Counterexample: $\sum \frac{(-1)^n}{n}$.

   (c) T If the series $\sum_{n=1}^{\infty} |b_n|$ converges, then the series $\sum_{n=1}^{\infty} b_n \cdot \sin n$ converges.

   $\sum_{n=1}^{\infty} b_n \cdot \sin n$ converges absolutely by Direct Comparison: $0 \le |b_n| \cdot |\sin n| \le |b_n|$.

   (d) T If the power series $\sum_{n=3}^{\infty} c_n x^n$ converges at $x = 3$, then it converges at $x = e$.

   Since the radius of convergence is at least 3.

   (e) ____ If the power series $\sum_{n=0}^{\infty} c_n x^n$ converges at $x = 3$, then it converges at $x = -3$.

   Counterexample: $\sum \frac{1}{(-3)^n} x^n$.

2. Discuss convergence of the following series: determine whether each is absolutely convergent, conditionally convergent, or divergent.

(a) (6 points) $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n^2 \cdot 3^n}{4^n}$.

The series converges absolutely by the Ratio Test, since

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left( \left|\frac{(n+1)^2 \cdot 3^{n+1}}{4^{n+1}}\right| \Big/ \left|\frac{n^2 \cdot 3^n}{4^n}\right| \right)$$
$$= \lim_{n\to\infty} \left|(\frac{n+1}{n})^2 \cdot \frac{3}{4}\right| = \frac{3}{4},$$

and $3/4 < 1$.

(b) (7 points) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}+2}$.

The series does not absolutely converge by Direct Comparison, since

$$\left|\frac{(-1)^n}{\sqrt{n}+2}\right| = \frac{1}{\sqrt{n}+2} \geq \frac{1}{3\sqrt{n}} \qquad \text{(for } n \geq 1\text{)}.$$

The terms are alternating, decreasing, and have limit zero, so the sequence does converge (conditionally), by the Alternating Series Test.

3. (8 points) Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{(n^2+1)\cdot 2^n}$.

As usual, we find the radius of convergence by the Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{\left((n+1)^2+1\right)\cdot 2^{n+1}}\right| \Big/ \left|\frac{x^n}{(n^2+1)\cdot 2^n}\right|$$
$$= \lim_{n\to\infty} \left|\frac{n^2+1}{n^2+2n+2}\right| \cdot \left|\frac{x}{2}\right|$$
$$= \lim_{n\to\infty} \left|\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}\right| \cdot \left|\frac{x}{2}\right| = \left|\frac{x}{2}\right|.$$

The series thus converges when $|x| < 2$ and diverges when $|x| > 2$, in short, has radius of convergence $R = 2$.

It remains to check the endpoints. Plugging in $\pm 2$ and canceling, we get the series $\sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^2+1}$. We get absolute convergence at both endpoints by Direct Comparison with $\sum \frac{1}{n^2}$, since $\frac{1}{n^2+1} < \frac{1}{n^2}$. Thus, the interval of convergence of the series is $[-2, 2]$.

4. Power series representations

(a) (6 points) Find a power series representation (around $a = 0$) for $\ln(x-2)$. At least 3 points will be given for a power series representation of $\dfrac{1}{x-2}$.

Note: there is a minor error in this problem. Since $\ln(-2)$ is not defined, it should have asked about a series for $\ln(2-x)$. The techniques involved are the same. First: we find a series for

$$\frac{1}{x-2} = -\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}} = -\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}}\cdot x^n,$$

by plugging in $\frac{x}{2}$ to the geometric series formula $\frac{1}{1-x}$.
(The series for $\frac{1}{2-x}$ differs only in sign.)

Then: we integrate to get $\ln(2-x)$:

$$\begin{aligned}\ln(2-x) &= \int \frac{-1}{2-x}\,dx = \int \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}}x^n\,dx \\ &= C + \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}}\cdot\frac{x^{n+1}}{n+1}.\end{aligned}$$

Finally: we solve for $C$.

$$\ln(2-0) = C + \sum_{n=0}^{\infty} 0 = C \qquad \Longrightarrow \qquad C = \ln 2.$$

Our final answer is

$$\ln(2-x) = \ln 2 + \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}\cdot(n+1)}x^{n+1}.$$

If desired, we could reindex this to (the equivalent answer)

$$\ln(2-x) = \ln 2 + \sum_{n=1}^{\infty} \frac{-1}{2^n\cdot n}x^n.$$

Note: A series for $\ln(x-2)$ proceeds in exactly the same way, except that you'd get $C = \ln(-2)$, which is not defined. Full credit was given for working through the problem and getting this “value” for $C$.

(b) (7 points) Find a power series representation (around $a = 0$) for $\dfrac{d^3}{dx^3}\left(\dfrac{1}{1-x^2}\right)$.

We first find a series for $\frac{1}{1-x^2}$ by plugging in to the geometric series formula:

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} \left(x^2\right)^k = \sum_{k=0}^{\infty} x^{2k}.$$

We then differentiate the series 3 times:

$$\begin{aligned}
\frac{d^3}{dx^3}\frac{1}{1-x^2} &= \frac{d^3}{dx^3}\sum_{k=0}^{\infty} x^{2k} \\
&= \frac{d^2}{dx^2}\sum_{k=1}^{\infty}(2k)x^{2k-1} \\
&= \frac{d}{dx}\sum_{k=1}^{\infty}(2k)(2k-1)x^{2k-2} \\
&= \sum_{k=2}^{\infty}(2k)(2k-1)(2k-2)x^{2k-3}.
\end{aligned}$$

Notice that the summation begins at $k = 1$ on the 2nd line, since the derivative of $x^0$ ($k = 0$) is 0. The summation also begins at $k = 1$ on the third line, since the first term in $\sum_{k=1}^{\infty}(2k)x^{2k-1}$ is $2x^1$. The summation begins at $k = 2$ in the final answer, since the first term of $\sum_{k=1}^{\infty}(2k)(2k-1)x^{2k-2}$ is $2 \cdot 1 \cdot x^0$.

5. (6 points) The "explain" problem.

Explain why $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ when $|r| < 1$.

Let $A_m$ be the partial sum $\sum_{n=0}^{m} r^n$. Then $A_{m+1} = A_m + r^{m+1} = rA_m + 1$. We solve for $A_m$ in this equation to get $(1-r) \cdot A_m = 1 - r^{m+1}$, hence $A_m = \dfrac{1-r^{m+1}}{1-r}$. We then have

$$\sum_{n=0}^{\infty} r^n = \lim_{m\to\infty} A_m = \lim_{m\to\infty} \frac{1-r^{m+1}}{1-r} = \frac{1}{1-r},$$

as desired.