MA 2733
Examination 2 Solutions – November 1, 2012

5 T/F, 3 long answer. 50 points.

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

   (a) __T__ The sequence $a_n = 4 \cdot 3^n$ can be expressed recursively as $a_1 = 12$, $a_n = 3a_{n-1}$ for $n > 1$.

   (b) __F__ $\sum_{n=1}^{\infty} a_n = L$ means that $\lim_{n\to\infty} \sum_{n=1}^{m} a_m = L$.

   While this looks somewhat similar to the definition of a series, it is nonsense. What is m here?

   (c) __F__ For the sequence $a_n = \sin(\frac{\pi}{2} n)$, we have $\lim_{n\to\infty} a_n = 0$.

   Since the sequence is $1, 0, -1, 0, 1, -1, \ldots$, and in particular is both 1 infinitely often and -1 infinitely often.

   (d) __T__ For the sequence $a_n = \frac{1}{n}\sin(n^2)$, we have $\lim_{n\to\infty} a_n = 0$.

   By the Squeeze Theorem, since $0 \leq \frac{1}{n}\sin(n^2) \leq \frac{1}{n}$.

   (e) __F__ The acceleration vector of any particle is orthogonal to its velocity vector.

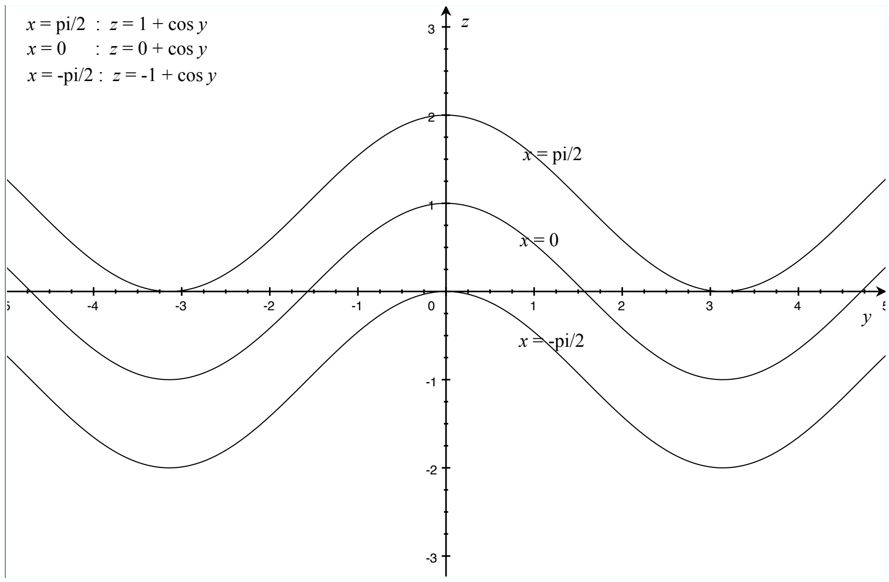
   Since $\mathbf{r}'' = a\mathbf{T} + b\mathbf{N}$, and the coefficient a of $\mathbf{T}$ may not be zero. The given statement is essentially only true for circular motion!

2. Planes and surfaces

   (a) (4 points) Find a vector that is orthogonal to the plane $x - y + z = 3$.

   The normal vector, $\langle 1, -1, 1\rangle$.

(b) (6 points) On the same axis, graph the traces of the surface $\sin x + \cos y = z$ in the planes $x = -\pi/2$, $x = 0$, and $x = \pi/2$.
For full credit: label each axis and indicate which trace is which!



3. (7 points) The "explain" problem.
Let $\mathbf{r}(t)$ be a vector function with $\|\mathbf{r}(t)\| = 3$. Show $\mathbf{r}(t)$ to be orthogonal to $\mathbf{r}'(t)$.

(At least 4 points will be given if you instead show $\mathbf{T}$ to be orthogonal to $\mathbf{T}'$.)

Since dot products detect orthogonality, it suffices to show that $\mathbf{r}\cdot\mathbf{r}' = 0$. Moreover

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = 2\mathbf{r}\cdot\mathbf{r}'$$

(by the chain rule or dot product rule), so it suffices to show that $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = 0$. But

$$\frac{dt}{dt}(\mathbf{r}\cdot\mathbf{r}) = \frac{d}{dt}(\|\mathbf{r}\|^2) = \frac{d}{dt}(3^2) = 0.$$

4. Calculus on vector functions.

(a) (15 points) Calculate the unit tangent vector $\mathbf{T}(t)$ and curvature $\kappa(t)$ for the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t^3/3\rangle$. We have $\mathbf{r}'(t) = \langle -\sin t, \cos t, t^2\rangle$, so that

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + t^4} = \sqrt{1+t^4},$$

and

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle -\sin t, \cos t, t^2\rangle}{\sqrt{1+t^4}}.$$

For $\kappa(t)$, we calculate $\mathbf{r}''(t) = \langle -\cos t, -\sin t, 2t\rangle$, thus that

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & t^2 \\ -\cos t & -\sin t & 2t \end{vmatrix} = \langle 2t\cos t + t^2 \sin t, -t^2\cos t + 2t\sin t, \sin^2 t + \cos^2 t\rangle.$$

Then

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{(2t\cos t + t^2\sin t)^2 + (-t^2\cos t + 2t\sin t)^2 + 1^2}}{(1+t^4)^{3/2}}.$$

The preceding got full credit, as I could plug a value of t into my \$5 calculator and find κ. For those who like to simplify, it simplifies (via the Pythagorean Theorem) to

$$\kappa(t) = \frac{\sqrt{4t^2 + t^4 + 1}}{(1+t^4)^{3/2}}.$$

(b) (8 points) Find the arc length function $s(t)$ of the curve $\mathbf{r}(t) = \langle \cos t^2, \sin t^2, t^3/3\rangle$.

Recall that $s(t)$ is the arc length of the curve between 0 and t.
(At least half credit is given if you just calculate $s(2\sqrt{3})$.) We first calculate

$\mathbf{r}'(t) = \langle -2t\sin t^2, 2t\cos t^2, t^2\rangle$, so that

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{4t^2\sin^2(t^2) + 4t^2\cos^2(t^2) + t^4} \\ &= \sqrt{4t^2 + t^4} \\ &= |t|\sqrt{4+t^2}. \end{aligned}$$

Then the arc length function is

$$s(t) = \int_0^t |x|\sqrt{4+x^2}\,dx.$$

Notice that we use "x" for the variable of integration, since we've already used t! We integrate by substituting $u = 4 + x^2$ so that $du = 2x\,dx$. For $t > 0$ this gives

$$\begin{aligned} s(t) &= \int_0^t x\sqrt{4+x^2}\,dx = \int_4^{4+t^2} \sqrt{u}\,\frac{du}{2} = \left[\frac{2}{3}u^{3/2}\cdot\frac{1}{2}\right]_4^{4+t^2} \\ &= \frac{1}{3}(4+t^2)^{3/2} - \frac{8}{3}. \end{aligned}$$

(If we plugged in $2\sqrt{3}$, we would get $56/3$.)