MA 2733

Examination 1 Solutions – October 2, 2012

5 T/F, 2 long answer. 50 points.

1. True/False. Enter T or F in each blank. A correct answer is worth 2 points, a blank space is worth 0 points, and a wrong answer is worth -2 points. (Your total on this problem will be rounded up to zero if necessary.)

   (a) T Every curve represented as a function $y = f(x)$ can also be represented parametrically.

   Taking $x = t$, $y = f(t)$.

   (b) T Every curve represented as in polar coordinates as $r = g(\theta)$ can also be represented parametrically.

   Taking $x = g(t)\cos t$, $y = g(t)\sin t$.

   (c) F Every curve represented parametrically as $x = r(t)$, $y = s(t)$ can also be represented as a function $y = f(x)$.

   Since the curve may fail the vertical line test.

   (d) T If $\mathbf{v}$, and $\mathbf{u}$ are vectors, then the triple product $(\mathbf{v} + \mathbf{u})\cdot(\mathbf{u} \times \mathbf{v})$ is always zero.

   Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ are coplanar; alternately, since

   $$(\mathbf{v} + \mathbf{u}) \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) + \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 + 0.$$

   (e) F If $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are vectors, then the triple product $\mathbf{w}\cdot(\mathbf{v} \times \mathbf{u})$ is always zero.

   Not every triple product is zero, consider e.g. $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

2. Parametric and polar equations

(a) (3 points) Convert the Cartesian (i.e., (x, y)) equation $y^2 = 2x^3 + 3x^2$ into polar coordinates.

Using $x = r\cos\theta$, $y = r\sin\theta$, we get

$$(r\sin\theta)^2 = 2(r\cos\theta)^3 + 3(r\cos\theta)^2.$$

(b) (5 points) Set up an integral for the arc length of the parametric curve given by $x = t^2$, $y = t^3$ for t on the interval $[0, 2]$. (You need not evaluate the integral.)

We have $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$, hence

$$L = \int_0^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt.$$

(c) (8 points) Find the area of one loop of the "8-leafed rose" given by the polar equation $r = \cos(4\theta)$.

We first find a loop of the 8-leafed rose. Since $r = \cos 4\theta = 0$ exactly when $\theta = \ldots, -\pi/8, \pi/8, 3\pi/8, \ldots$, we can take θ on the interval $[-\pi/8, \pi/8]$. Then we plug this into the polar area formula:

$$\begin{aligned} A &= \int_{-\pi/8}^{\pi/8} \frac{1}{2} r^2\, d\theta = \int_{-\pi/8}^{\pi/8} \frac{1}{2}\cos^2(4\theta)\, d\theta \\ &= \int_0^{\pi/8} \cos^2(4\theta)\, d\theta \qquad \text{(by symmetry)} \\ &= \int_0^{\pi/8} \frac{1+\cos(8\theta)}{2}\, d\theta \qquad \text{(by } \tfrac{1}{2}\text{-angle formula)} \\ &= \left[\frac{\theta}{2} + \frac{\sin(8\theta)}{16}\right]_0^{\pi/8} = \frac{\pi}{16}. \end{aligned}$$

(d) (8 points) Find all real values of t at which the parametric curve given by $x = t^2$, $y = 3t^2 - 2t^3$ has a horizontal tangent line.

We calculate

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{6t - 6t^2}{2t}.$$

Except at $t = 0$, this is equal to $3 - 3t$, which has a root at 1. Since horizontal tangents correspond to points with 0 derivative, the curve has a horizontal

tangent line exactly when $t = 1$.
(Note that the tangent line when $t = 0$ has slope 3.)

3. Vectors and straight-line geometry

   (a) (8 points) Give the vector equation of a straight line through the origin which is orthogonal to both of the straight lines $\mathbf{r} = t\langle 1, 2, 3\rangle$ and $\mathbf{r} = t\langle 1, 1, -1\rangle$. (At least half credit is given for finding a vector orthogonal to both lines.)

   A vector orthogonal to both lines is $\langle 1, 2, 3\rangle \times \langle 1, 1, -1\rangle = \langle -5, 4, -1\rangle$. A line through the origin in the direction of this vector is given by the equation $\mathbf{r} = t\langle -5, 4, -1\rangle$.

   (b) (2 points) In 1-3 sentences, explain why dot products and cross products are important for geometry in 3 dimensions.

   Various answers are possible: e.g. dot products help us detect whether two vectors are orthogonal, while cross products allow us to create a vector that is orthogonal to two given vectors.
   Your answer should give some specific reasons, and should mention angles, orthogonality, volume, or similar.

   (c) (6 points) Explain why $\text{proj}_{\mathbf{u}}\,\mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}$. You may use the fact that $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$. A clear picture is by itself worth at least 2 points.

   By the triangle pictured,

$$\|\text{proj}_u\,\mathbf{v}\| = \|\mathbf{v}\|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}.$$

   A vector of length 1 with the same direction as $\mathbf{u}$ is $\dfrac{\mathbf{u}}{\|\mathbf{u}\|}$. Therefore,

$$\text{proj}_u\,\mathbf{v} = \|\text{proj}_u\,\mathbf{v}\|\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}\frac{\mathbf{u}}{\|\mathbf{u}\|},$$

   as desired.