Weighted bond posets and graph associahedra

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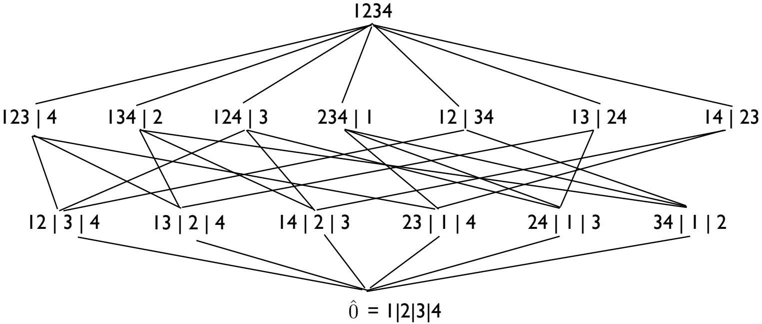
Joint work with Rafael González D'León

Partition lattice Π_n

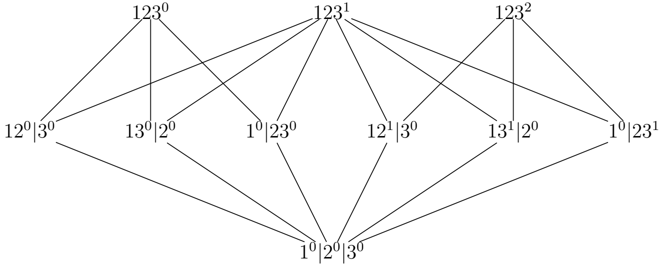


Classical results:

- Π_n is EL-shellable.
- $\Delta(\bar{\Pi}_n)$ has homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$.
- $$\tilde{H}_{n-3}(\bar{\Pi}_n) \cong_{\mathfrak{S}_n} Lie_n \otimes sgn_n,$$

  where Lie_n is the multilinear component of the free Lie algebra with n generators.

Weighted partition poset Π_n^w (Dotsenko & Khoroshkin (2007))



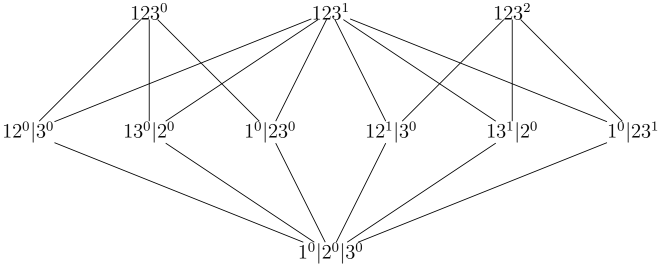
A weighted partition of $[n]$ is a set $\{B_1^{v_1}, \ldots, B_k^{v_k}\}$ where

- $\{B_1, \ldots, B_k\} \in \Pi_n$
- $v_i \in \mathbb{N}$ and $0 \leq v_i \leq |B_i| - 1 \ \forall i$.

Cover relation: $\{A_1^{u_1}, \ldots, A_j^{u_j}\} \lessdot \{B_1^{v_1}, \ldots, B_k^{v_k}\}$ if

- $\{A_1, \ldots, A_j\} \lessdot \{B_1, \ldots, B_k\}$ in Π_n.
- $B_r = A_s \cup A_t$ with $s \neq t \implies v_r - (u_s + u_t) \in \{0, 1\}$
- $B_r = A_s \implies v_r = u_s$.

Weighted partition poset Π_n^w (Dotsenko & Khoroshkin (2007))



unique minimum: $\hat{0} = 1^0|2^0|\ldots|n^0$
n maximal elements: $[n]^0, [n]^1, \ldots, [n]^{n-1}$

Note:

- $[\hat{0}, [n]^j] \cong [\hat{0}, [n]^{n-1-j}]$ for all $j = 0, \ldots, n-1$
- $[\hat{0}, [n]^0] \cong \Pi_n$

Properties of maximal intervals $[\hat{0}, [n]^j]$

González D'León and MW (2012):

- Each $[\hat{0}, [n]^j]$ is EL-shellable.
- $\Delta((\hat{0}, [n]^j))$ has the homotopy type of $\beta_{n,j}$ $(n-3)$-spheres, where $\beta_{n,j}$ is the the number of rooted trees on $[n]$ with j descents.
- Explicit $\mathfrak{S}_n$-module isomorphism:

$$\tilde{H}_{n-3}((\hat{0}, [n]^j)) \cong_{\mathfrak{S}_n} Lie^2_{n,j} \otimes sgn_n,$$

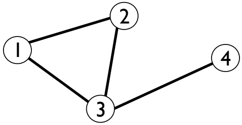
  where $Lie^2_{n,j}$ is the multilinear component of the free Lie algebra with two compatible brackets with one bracket occurring j times and the other occuring $n-1-j$ times.

González D'León (2013) introduced a more general weight in order to study the free Lie algebra with more than two compatible brackets.

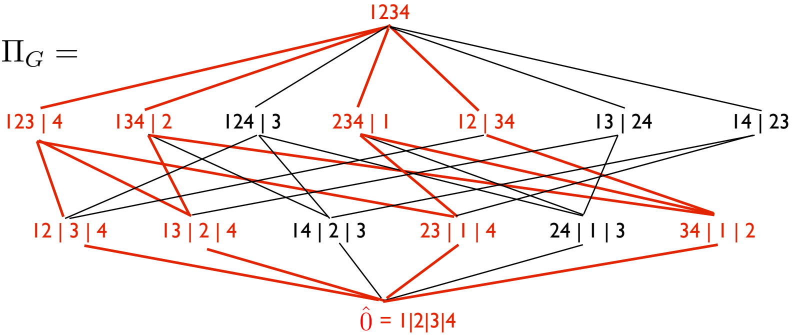
The bond lattice of a graph

The bond lattice Π_G of a graph $G = ([n], E)$ is the induced subposet of Π_n consisting of the partitions whose blocks are vertex sets of connected subgraphs of G.

$G =$



$\Pi_G =$



The weighted bond poset

Let Π_G^w be the induced subposet of Π_n^w consisting of the weighted partitions whose blocks are vertex sets of connected subgraphs of G.

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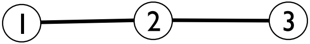

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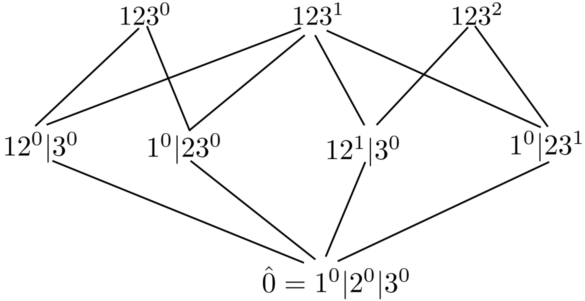

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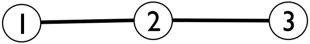

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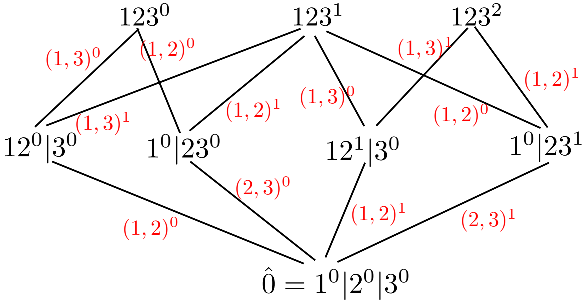


The EL-labeling of Π_n^w restricted to Π_G^w

EL-shellability of the weighted bond poset

A graph is said to be chordal if it has no induced cycles of length greater than 3.

Examples:

- complete graph
- path $1 - 2 - \cdots - n$
- tree

Theorem (González D'León and MW)

If G is a chordal graph then each interval of Π_G^w is EL-shellable.

To prove this we use the fact that every chordal graph is isomorphic to a graph on $[n]$ that has no induced paths $i - k - j$ in G, where $i < j < k$. Call such graphs 132-free. We show that the EL-labeling that we used for Π_n^w works for Π_G^w whenever G is 132-free.

Betti number polynomial

Let G be connected and chordal. Since each maximal interval $[\hat{0}, [n]^j]$ of Π_G^w is EL-shellable, it has the homotopy type of a wedge of $\beta_{G,j}$ $(n-3)$-spheres.

Define the Betti number polynomial

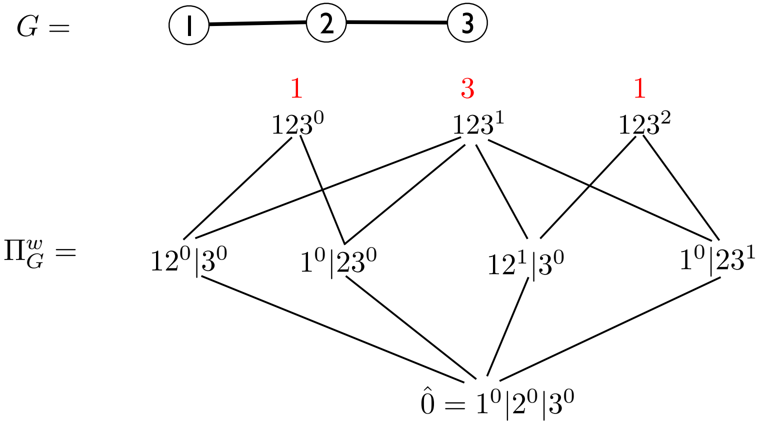
$$\beta_G(t) = \sum_{j=0}^{n-1} \beta_{G,j} t^j.$$

By counting the ascent-free chains of the EL-labeling we obtain the following:

Corollary (González D'León and MW)

Let G be connected and chordal. Then $\beta_G(t)$ is γ-positive. Consequently $\beta_G(t)$ is palindromic and unimodal.

Betti number polynomial $\beta_G(t) = \sum_{j=0}^{n-1} \beta_{G,j} t^j$



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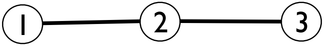

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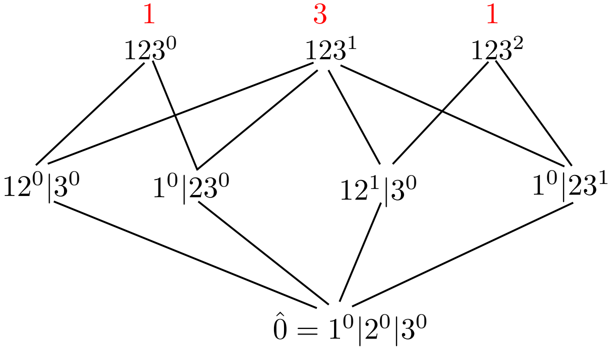

$$\beta_{G_3}(t) = 1 + 3t + t^2$$

$$\beta_{G_4}(t) = 1 + 6t + 6t^2 + t^3$$

Betti number polynomial $\beta_G(t) = \sum_{j=0}^{n-1} \beta_{G,j} t^j$

Theorem (González D'León and MW)

Let G_n be the path on $[n]$. Then $\beta_{G_n}(t)$ is the Narayana polynomial of degree $n-1$.

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Let G_n be the path on $[n]$. Then $\beta_{G_n}(t)$ is the Narayana polynomial of degree $n-1$.

It is well-known that the Narayana polynomial is the h-polynomial of the associahedron. So the result says that the Betti polynomial for the path equals the h-polynomial for the associahedron.

Recall the h-polynomial of a simple d-dimensional polytope is defined by

$$\sum_{k=0}^{d} h_k t^k = \sum_{i=0}^{d} f_i (t-1)^i,$$

where f_i is the number of faces of dimension i.

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Can this result be generalized to other graphs? Are there other graphs whose Betti number polynomial is the h-polynomial of a polytope?

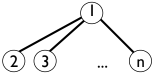
Betti number polynomial $\beta_G(t) = \sum_{j=0}^{n-1} \beta_{G,j} t^j$

Conjecture (González D'León and MW)

Let T be a tree on node set $[n]$. Then $\beta_T(t)$ is the h-polynomial of a simple polytope, namely the graph associahedron associated with T.

True for

- the path
- the star graph



- $n \leq 9$.

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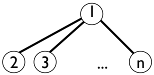


- $n \leq 9$.

Postnikov, Reiner, and Williams showed that the h-polynomial of any chordal graph associahedron is γ-positive.

Graph associahedra

Discovered independently by

- Carr and Devadoss (2004)
- Davis, Januszkiewicz, and Scott (2003)
- Postnikov (2005)

Connections to

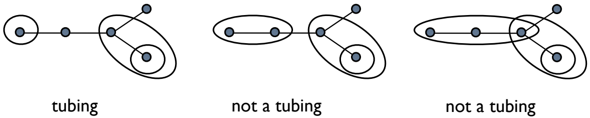
- Delign-Knudsen-Mumford compactification of real moduli space of curves
- De Concini-Procesi "wonderful" models of hyperplane arrangements
- Bergman complexes of oriented matroids
- graphical tests in statistics

Graph associahedra

Let $G = ([n], E)$ be connected. A tube is a proper nonempty subset of $[n]$ that forms the vertex set of a connected subgraph of G. We say that two tubes u_1 and u_2 are

- intersecting if $u_1 \cap u_2 \neq \emptyset$ and neither contains the other
- adjacent if $u_1 \cap u_2 = \emptyset$ and $u_1 \cup u_2$ is a tube.

A tubing of G is a set of tubes of G no two of which intersect or are adjacent.



The graph associahedron $\mathcal{A}_G$ is a simple convex polytope whose face poset is isomorphic to the poset of tubings of G ordered by reverse containment.

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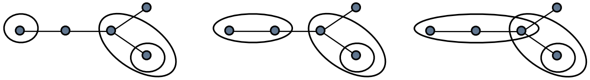


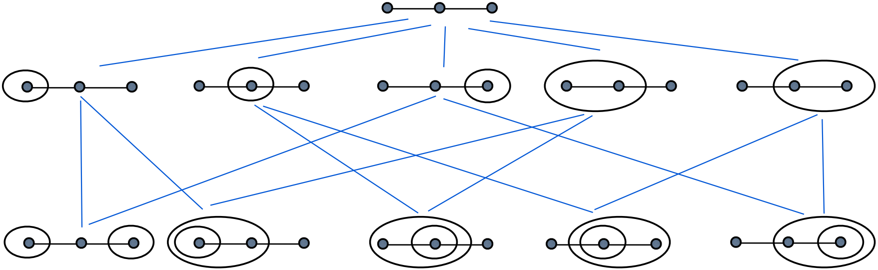
tubing　　not a tubing　　not a tubing

The graph associahedron $\mathcal{A}_G$ is a simple convex polytope whose face poset is isomorphic to the poset of tubings of G ordered by reverse containment.

Carr and Devados give a nice realization of the polytope obtained by truncating the faces of a simplex that correspond to the tubes of G in increasing order of dimension.

Graph associahedron for the path G_3

$G_3 =$ 



$\cong$ face poset of $\mathcal{A}_{G_3}$

$\mathcal{A}_{G_3} =$ 

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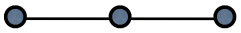
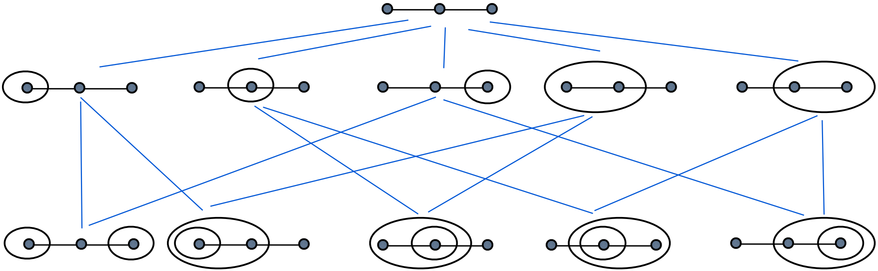
$G_3 =$ 



$\cong$ face poset of $\mathcal{A}_{G_3}$

$\mathcal{A}_{G_3} =$ 

$$h(t) = 5 + 5(t-1) + (t-1)^2 = 1 + 3t + t^2 = \beta_{G_3}(t)$$