On some S_n-modules induced from centraliser subgroups

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1. *The multiplicity of the irreducible χ equals the sum $\sum_C \chi(C)$ where C ranges over all the conjugacy classes of G.*
2. *Its character is given by the formula $\sum_{\chi \in Irr(G)} \chi\bar{\chi}$, where $Irr(G)$ is the set of irreducible characters of G.*

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$$g\sigma g^{-1} = \sigma \iff g\sigma = \sigma g \iff \sigma \in Z(g).$$

Partitions of n and symmetric functions

- $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \geq 1)$ such that $\sum_i \lambda_i = n$ is an integer partition of n; $\ell(\lambda)$ is the number of parts k of λ.

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- Up to a scalar multiple, p_λ is the Frobenius characteristic of the class function that is 1 on the conjugacy class indexed by λ and zero elsewhere;
- s_λ is the Frobenius characteristic of the S_n-irreducible indexed by λ.

The symmetric group

For the symmetric group S_n, Solomon's result says (cf. Richard Stanley's EC2, Exercise 7.71):

Theorem

The permutation representation of S_n acting on itself by conjugation has Frobenius characteristic given by

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Implications for the character table of G

Corollary

The row-sums in the character table of G (indexed by the irreducible characters of G, or equivalently by the conjugacy classes of G) are nonnegative integers.

Character tables of S_n

	$\underline{2}$	$\underline{1^2}$	ROW SUM
(2)	1	1	2
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(3)	1	1	1	3
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	$\underline{4}$	$\underline{3,1}$	$\underline{2^2}$	$\underline{2,1^2}$	$\underline{1^4}$	ROW SUM
(4)	1	1	1	1	1	5
$(3,1)$	-1	0	-1	1	3	2
(2^2)	0	-1	2	0	2	3
$(2,1^2)$	1	0	-1	-1	3	2
(1^4)	-1	1	1	-1	1	1

Questions addressed in this talk

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Equivalently:

Question

Which subsets T_n of the set of partitions of n have the property that the sum of power-sum symmetric functions

$$\sum_{\lambda \in T_n} p_\lambda$$

is Schur-positive, i.e., arises as the Frobenius characteristic of an S_n-module?

A simple observation

If the subset T_n does not contain (1^n) then the sum

$$\sum_{\lambda \in T_n} p_\lambda$$

cannot be the Frobenius characteristic of a true module.

1. $T_6 = \{(1^6), (2, 1^4), (3^2), (4, 2), (4, 1^2)\}$: the multiplicity of the Schur function indexed by $(2, 1^4)$ in
$$p_1^6 + p_2 p_1^4 + p_3^2 + p_4 p_2 + p_4 p_1^2$$
is -1;

Some examples

1. $T_6 = \{(1^6), (2, 1^4), (3^2), (4, 2), (4, 1^2)\}$: the multiplicity of the Schur function indexed by $(2, 1^4)$ in
$$p_1^6 + p_2 p_1^4 + p_3^2 + p_4 p_2 + p_4 p_1^2$$
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2. $T_6 = \{(1^6), (2^3), (3, 1^3), (4, 1^2), (4, 2)\}$: the multiplicity of the Schur function indexed by (3^2) in
$$p_1^6 + p_2^3 + p_3 p_1^3 + p_4 p_1^2 + p_4 p_2$$
is -1.

Sample results

Let Par_n denote the set of all partitions of n, and DO_n the set of all partitions of n with distinct odd parts.

Definition

For a subset T_n of the partitions of n, define $P_{T_n} = \sum_{\lambda \in T_n} p_\lambda$.

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Theorem

(S, 2015) The symmetric function P_{T_n} is Schur-positive when:

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4. $T_n = \{\lambda \in Par_n : \lambda \notin DO_n\}, n \geq 2$.
5. $T_n = \{\lambda \in Par_n : \lambda_i = 1 \textit{ or } k\}$, *for any fixed* $k \geq 1$.

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5. $T_n = \{\lambda \in Par_n : \lambda_i = 1 \textit{ or } k\}$, *for any fixed* $k \geq 1$.
6. $T_n = \{\lambda \in Par_n : \lambda_i \textit{ divides } k\}$, *for any fixed* $k \geq 1$.

More examples

Including the partition (1^n) does not guarantee Schur-positivity. Suppose T_n consists of (1^n) and all partitions λ of n such that $n - \ell(\lambda)$ is odd.

- $P_{T_4} = p_1^4 + p_2 p_1^2 + p_4 = 3s_{(4)} + 3s_{(3,1)} + 2s_{(2^2)} + 3s_{(2,1^2)} - s_{(1^4)}$
- $P_{T_5} = p_1^5 + p_2 p_1^3 + p_3 p_2 + p_4 p_1$
  $= 4s_{(5)} + 5s_{(4,1)} + 6s_{(3,2)} + 6s_{(3,1^2)} + 4s_{(2^2,1)} + 3s_{(2,1^3)} - 2s_{(1^5)}$
- $P_{T_6} = p_1^6 + p_2 p_1^4 + p_2^3 + p_3 p_2 p_1 + p_4 p_1^2 + p_6$
  $= 6s_{(6)} + 7s_{(5,1)} + 14s_{(4,2)} + 10s_{(4,1^2)} + 3s_{(3^2)} + 16s_{(3,2,1)} + 10s_{(3,1^3)}$
  $+ 7s_{(2^3)} + 4s_{(2^2,1^2)} + 3s_{(2,1^4)} - 4s_{(1^6)}$

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- These are the conjugacy classes, hence for S_n, are indexed by partitions of n.
- Conjugation transitively permutes the n-cycles, so on the class of n-cycles it is the trivial representation induced from a cyclic subgroup C_n of order n.
- For arbitrary λ, if σ is a permutation of type λ, the conjugation action is the trivial representation induced from the centraliser of σ.

If λ has m_i parts equal to i, the centraliser of an element with cycle-type λ is the **wreath product**

$$S_{m_1}[C_1] \times S_{m_2}[C_2] \times \ldots \times S_{m_i}[C_i] \times \ldots$$

And wreath products lead to ...

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Plethysm of symmetric functions:

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Let f_n be the Frobenius characteristic of the conjugacy action on the n-cycles. Then the Frobenius characteristic of the conjugacy action on an element of cycle-type λ is

$$H_\lambda[F] = h_{m_1}[f_1] \times h_{m_2}[f_2] \times \ldots \times h_{m_i}[f_i] \times \ldots$$

where $g[f]$ is the plethysm operation, and h_n is the homogeneous symmetric function.

Power-sum decomposition for the conjugacy action of S_n

Theorem

(S, 2015) $\sum_{\lambda \vdash n} H_\lambda[F] = \sum_{\mu \vdash n} p_\mu$.

A twisted conjugacy action

Definition

(S) Define an action of S_n on elements of a fixed cycle-type as follows: The stabiliser $S_k[C_m]$ acts trivially within a cycle, and acts on the m-cycles according to the sign representation of S_k. Inducing from each centraliser up to S_n gives a twisted analogue of the conjugation action.

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Proposition

(S, 2015) The Frobenius characteristic of the S_n-representation on the orbit indexed by λ is

$$E_\lambda[F] = \prod_i e_{m_i}[f_i],$$

where e_n is the nth elementary symmetric function, and λ has m_i parts equal to i.

Theorem

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Hence

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Hence

- *the sum on the right is Schur-positive;*
- *the twisted conjugacy representation is self-conjugate.*

Two special submodules of the conjugacy action by parity

Consider the alternating subgroup A_n of S_n, and its complement $\bar{A}_n$. Define $\psi(S_n, A_n)$ and $\psi(S_n, \bar{A}_n)$ to be the conjugacy actions on A_n and its complement respectively.

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If $\psi(S_n)$ denotes the conjugacy action of S_n on itself, we have

$$\psi(S_n) = \psi(S_n, A_n) \oplus \psi(S_n, \bar{A}_n).$$

Each summand is a permutation module of dimension $n!/2$.

Power-sum decompositions for actions on even and odd permutations

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Similar results for the twisted conjugacy action

Let $Par_n^{\neq}$ be the set of partitions of n with distinct parts.

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3. $ch\,\varepsilon(S_n, \bar{A}_n) = \frac{1}{2}\left(\sum_{\lambda \in Par_n^{odd}} p_\lambda - \sum_{\lambda \in Par_n^{\neq}} \omega(p_\lambda)\right)$

	$\underline{2}$	$\underline{1^2}$	Row Sum	Odd Row Sum
(2)	1	1	**2**	**1**
(1^2)	-1	1	**0**	**1**

	$\underline{3}$	$\underline{2,1}$	$\underline{1^3}$	Row sum	Odd row sum
(3)	1	1	1	**3**	**2**
$(2,1)$	-1	0	2	**1**	**1**
(1^3)	1	-1	1	**1**	**2**

Character tables for S_4

	$\underline{4}$	$\underline{3,1}$	$\underline{2^2}$	$\underline{2,1^2}$	$\underline{1^4}$	Row sum	Odd row sum
(4)	1	1	1	1	1	**5**	**2**
$(3,1)$	-1	0	-1	1	3	**2**	**3**
(2^2)	0	-1	2	0	2	**3**	**1**
$(2,1^2)$	1	0	-1	-1	3	**2**	**3**
(1^4)	-1	1	1	-1	1	**1**	**2**

Character tables: S_5

	$\underline{5}$	$\underline{4,1}$	$\underline{3,2}$	$\underline{3,1^2}$	$\underline{2^2,1}$	$\underline{2,1^3}$	$\underline{1^5}$	RowΣ	OddΣ
(5)	1	1	1	1	1	1	1	**7**	**3**
$(4,1)$	-1	0	-1	1	0	2	4	**5**	**4**
$(3,2)$	0	-1	1	-1	1	1	5	**6**	**4**
$(3,1^2)$	1	0	0	0	-2	0	6	**5**	**7**
$(2^2,1)$	0	1	-1	-1	1	-1	5	**4**	**4**
$(2,1^3)$	-1	0	1	1	0	-2	4	**3**	**4**
(1^5)	1	-1	-1	1	1	-1	1	**1**	**3**

	C^1	$\mathbf{C^2}$	C^3	C^4	$\mathbf{C^5}$	C^6	$\mathbf{C^7}$	C^8	C^9	C^{10}	$\mathbf{C^{11}}$	R	**O**
(6)	1	1	1	1	1	1	1	1	1	1	1	**11**	**4**
$(5,1)$	−1	0	−1	1	−1	0	2	−1	1	3	5	**8**	**6**
$(4,2)$	0	−1	1	−1	0	0	0	3	1	3	9	**15**	**8**
$(4,1^2)$	1	0	0	0	1	−1	1	−2	−2	2	10	**10**	**12**
(3^2)	0	0	−1	−1	2	1	−1	−3	1	1	5	**4**	**6**
$(3,2,1)$	0	1	0	0	−2	0	−2	0	0	0	16	**13**	**13**
$(3,1^3)$	−1	0	0	0	1	1	1	2	−2	−2	10	**10**	**12**
(2^3)	0	0	−1	1	2	−1	−1	3	1	−1	5	**8**	**6**
$(2^2,1^2)$	0	−1	1	1	0	0	0	−3	1	−3	9	**5**	**8**
$(2,1^4)$	1	0	−1	−1	−1	0	2	1	1	−3	5	**4**	**6**
(1^6)	−1	1	1	−1	1	−1	1	−1	1	−1	1	**1**	**4**

Character tables: S_6

Irrep	C^1	$\underline{C^2}$	C^3	C^4	$\underline{C^5}$	C^6	$\underline{C^7}$	C^8	C^9	C^{10}	$\underline{C^{11}}$	R	$\underline{\text{O}}$
(6)	1	1	1	1	1	1	1	1	1	1	1	**11**	**4**
(5,1)	-1	0	-1	1	-1	0	2	-1	1	3	5	**8**	**6**
(4,2)	0	-1	1	-1	0	0	0	3	1	3	9	**15**	**8**
$(4, 1^2)$	1	0	0	0	1	-1	1	-2	-2	2	10	**10**	**12**
(3^2)	0	0	-1	-1	2	1	-1	-3	1	1	5	**4**	**6**
(3,2,1)	0	1	0	0	-2	0	-2	0	0	0	16	**13**	**13**
$(3, 1^3)$	-1	0	0	0	1	1	1	2	-2	-2	10	**10**	**12**
(2^3)	0	0	-1	1	2	-1	-1	3	1	-1	5	**8**	**6**
$(2^2, 1^2)$	0	-1	1	1	0	0	0	-3	1	-3	9	**5**	**8**
$(2, 1^4)$	1	0	-1	-1	-1	0	2	1	1	-3	5	**4**	**6**
(1^6)	-1	1	1	-1	1	-1	1	-1	1	-1	1	**1**	**4**

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Thomas Scharf (1990) gave another proof using the fact that the conjugacy action can be computed as a plethysm.

Which representations contain every irreducible?

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(S, 2015) The following representations contain every S_n-irreducible:

1. $\psi(S_n, A_n)$ *for* $n \geq 4$. *(Missing irreducibles are* (1^2) *and* $(2,1)$*.)*

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(S, 2015) The row-sums in the character table of S_n are strictly positive when restricted to columns indexed by the following subsets T_n of partitions:

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4. $T_n = \{\lambda \in Par_n : \lambda \notin DO_n\}, n \geq 2$, with an exception for the row-sum indexed by the sign character.

The alternating group

Consider A_n acting on itself by conjugation. By inducing this action up to S_n, we have yet another S_n-module $\psi(A_n) \uparrow_{A_n}^{S_n}$ of dimension $n!$.

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- $= \sum_{\lambda \in DO_n} p_\lambda + \sum_{\substack{\mu \in Par_n \\ n-\ell(\mu)\ even}} p_\mu$.

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For $G = S_n$ this was first proved by Avital Frumkin; for $G = A_n$ it is a consequence of work of Heide-Saxl-Thiep-Zalessky (or in Heide's Ph.D. thesis?)

Can define analogues of $\psi(S_n, A_n)$ and $\psi(S_n, \bar{A}_n)$.

Proposition

(S, 2015) Let G be a group with a subgroup H of index 2. Let W be any representation of H. Then W contains every H-irreducible if and only if the induced action $W \uparrow_H^G$ contains every G-irreducible.

The work of Heide-Saxl-Thiep-Zalessky

Theorem

(Heide-Saxl-Thiep-Zalessky, 2013) If G is a finite simple group of Lie type other than $PSU_n(q)$ with $n \geq 3$ coprime to $2(q+1)$, then the conjugacy action of G contains every complex G-irreducible. (In the exceptional case only one irreducible is missing.)

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2. How close or how far away are these representations from the regular representation?