LCM lattices of pure resolutions

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(with C. Francisco and J. Mermin, Oklahoma State University)

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- Often we just care about the degrees of these shifting monomials, so

$$F_i = \bigoplus_j S[-j]^{a_{i,j}}$$

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- So if the resolution is pure, then, for each i, $b_{i,j} \neq 0$ for only one value of j.

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- Let I be a (not necessarily squarefree) monomial ideal with minimal generating set $X = \{m_1, m_2, \ldots, m_t\}$.
- The *LCM*-lattice of I is the lattice of all lcm's of subsets of X, ordered by divisibility.

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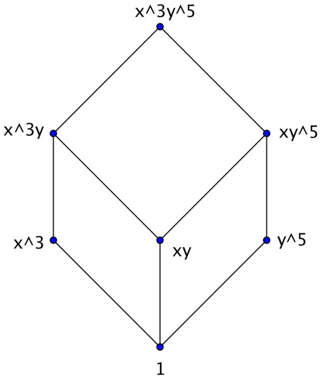

- Let P be a poset. If $a < b$ in P, we write (a, b) for the *order complex* of $[a, b]$, the simplicial complex whose faces are chains in between a and b (but not including a and b).

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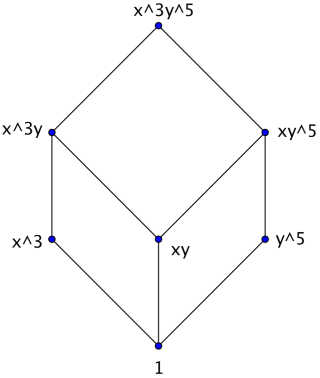
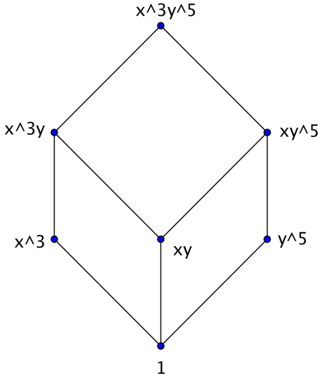
Theorem (Gasharov, Peeva, Welker)

The Betti numbers of a monomial ideal I are given by

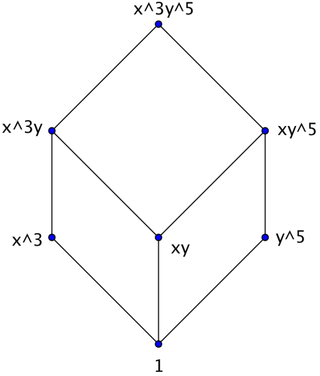
$$b_{i,m} = \text{rank}(\tilde{H}_{i-2}(1, m))$$

Where $m \neq 1$.


x^3y^5
x^3y
xy^5
x^3
xy
y^5
1



- So $b_{1,x^3} = \text{rank}(\tilde{H}_{-1}(1, x^3)) = 1$



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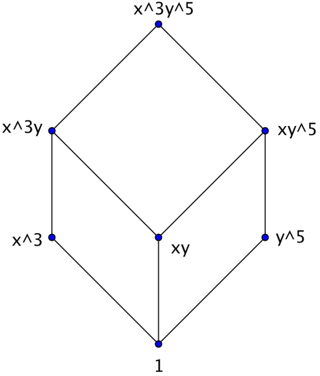

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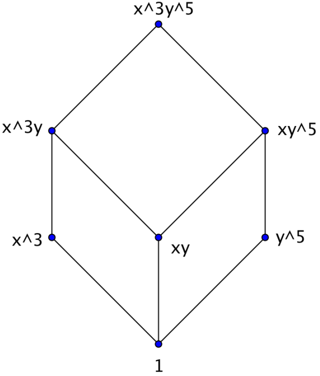

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- So $b_{2,4} = b_{2,6} = b_{1,3} = b_{1,2} = b_{1,5} = 1$. Note that I does not have a pure resolution.

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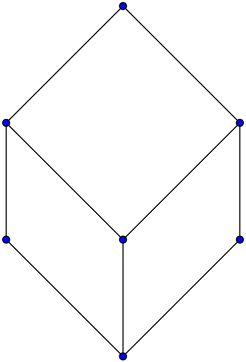

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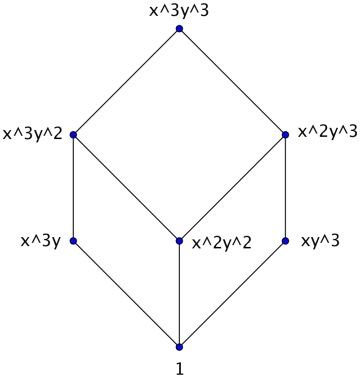

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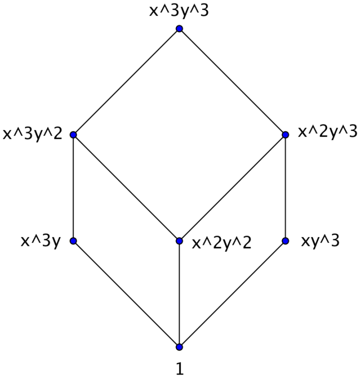

- Here, $b_{1,3} = 3$, $b_{2,5} = 2$, all other $b_{i,j} = 0$.

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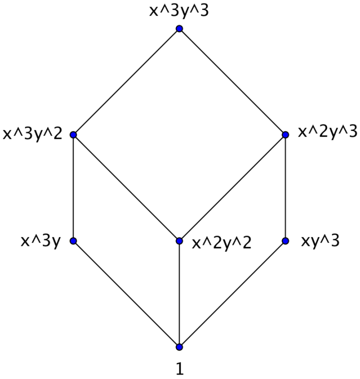

- Here, $b_{1,3} = 3$, $b_{2,5} = 2$, all other $b_{i,j} = 0$. So, the associated ideal is pure.

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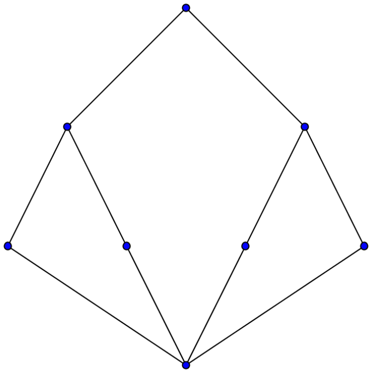

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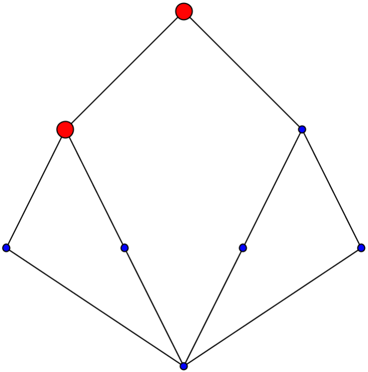

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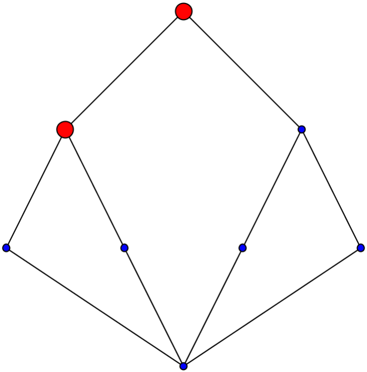

- Is this an LCM lattice, though?

Theorem (Mapes, Phan)

Every atomic lattice is an LCM lattice.

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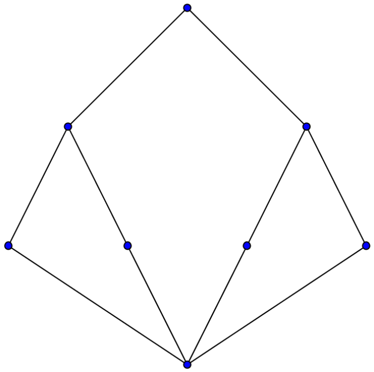

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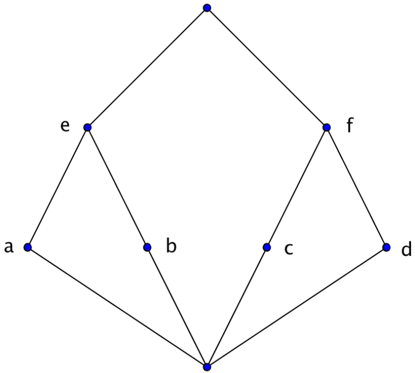

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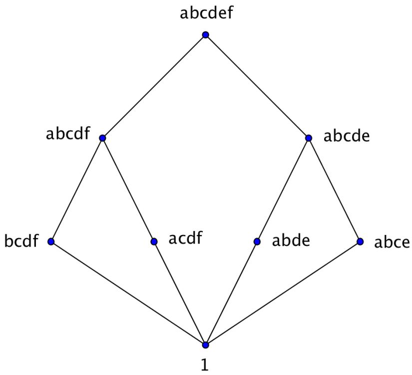

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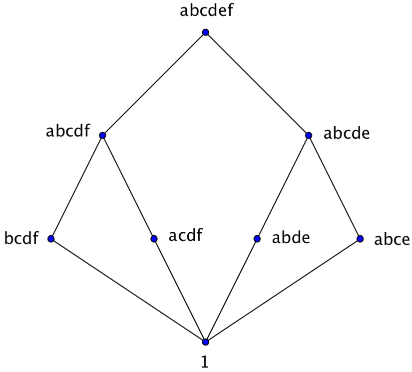

- Note that $b_{2,abcdef} = 1 = b_{2,abcdf}$, so $b_{2,5} \neq 0 \neq b_{2,6}$.

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Call a lattice L *homologically monotonic* or *HM* if whenever $x < y$ in L and

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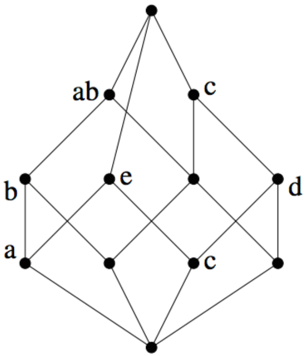
- Idea of proof: Start with an ideal I whose LCM lattice is L. Look at the highest i for which some $b_{i,j}$ and $b_{i,k}$ are both nonzero (i.e., the largest i at which pureness fails). Modify the "tags" of the lattice so that the resultant ideal is pure in the ith place, and such that pureness isn't messed up for higher indices.


ab
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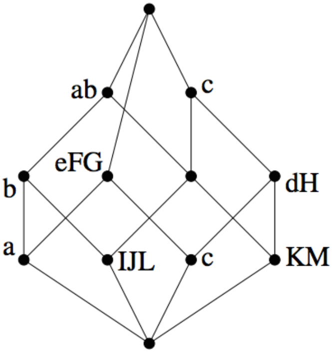
ab
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- Is there a way to begin with an HM lattice, and immediately construct a pure ideal I whose LCM lattice is L?
- Are there "natural" ideals with pure resolutions that are associated to particularly nice lattices? (i.e., geometric, supersolvable, or more generally EL-shellable)

THANKS!!