The topology of the external activity complex of a matroid

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October 18, 2015

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Outline

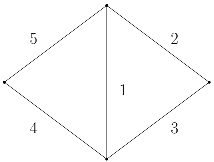
- Motivation and setup
- The external activity complex and shellability
- Consequences and conclusion

Ordered matroids

- **Matroid:** M with ground set E. Let $\mathcal{B}$ be the set of *bases*, $IN(M)$ the *independence complex* and $\mathcal{C}$ the set of *circuits*.

- **Ordered matroid:** $(M, <)$ where $<$ is a total order on E. Used a lot in the literature (shellings, nbc complexes, basis for the Orlik-Solomon algebra etc)!!!

Example



- $E = [5]$ are the edges of the graph.
- $\mathcal{B} = \{124, 125, 134, 135, 234, 235, 245, 345\}$ spanning trees.
- $\mathcal{C} = \{123, 145, 2345\}$.
- Notice that it comes with an obvious order!

Motivation

We are going to study a complex associated to matroid called $Act<(M)$

- It shows up when studying the coordinate ring of the closure of a linear space in a product of projective lines.
- Is related to the reciprocal plane studied by Proudfoot and Speyer (for realisable matroids).
- It seems suitable to attack a classical conjecture of Stanley: the h-vector of the independence complex of a matroid is a pure O-sequence.

More on oredered matroids

- $(M, <)$ ordered matroid.
- $B \in \mathcal{B}$ and $b \in E - B$, there is a unique circuit $CI(B, b)$ contained in $B \cup \{b\}$.
- If $b = \min CI(B, b)$ we say that b is *externally active* wrt B. b is *externally passive* otherwise.
- $EA(B)$ and $EP(B)$ denote the sets of externally active and passive elements.
- The dual notions of external activities are called *internal activities* and dented by $IA(B)$ and $IP(B)$. Related to the shelling of the matroid
- The **broken circuit** $bc(C)$ of a circuit C is the set $C \backslash c$, where c is the smallest element of C, denoted by $\min C$.

example

B	$EP(B)$	$EA(B)$	$IP(B)$	$IA(B)$
124	35	$\emptyset$	$\emptyset$	124
125	34‘	$\emptyset$	5	12
134	25	$\emptyset$	3	14
135	24	$\emptyset$	35	1
234	5	1	23	4
235	4	1	235	$\emptyset$
245	3	1	45	2
345	$\emptyset$	12	345	$\emptyset$

Table : Recall that $\mathcal{C} = \{123, 145, 2345\}$

The external activity complex

$(M,<)$ matroid with goundset E. Let $\overline{E}$ be a disjoint copy of E. For $e \in E$ denote by $\overline{e}$ its copy in $\overline{E}$ and let $[E] = E \cup \overline{E}$.

Theorem (Ardila-Bocher 13)

There is a simplicial complex $Act < (M)$ on the vertex set $[E]$ such that:

- Its minimal non-faces are of the form $S(C) := \min C \cup \overline{bc(C)}$ for every circuit $C \in \mathcal{C}$.
- Its facets are the sets $F(B) := B \cup EP(B) \cup \overline{B \cup EA(B)}$.
- Its algebraic Betti numbers are equal to the ones of $IN(M)$, thus it is Cohen-Macualay and has the same h-vector as $Act_<(M)$.

Example

B	$EP(B)$	$EA(B)$	$F(B)$	$\hat{F}(B)$
124	35	$\emptyset$	$12345\overline{124}$	$12\overline{24}$
125	34	$\emptyset$	$12345\overline{125}$	$12\overline{25}$
134	25	$\emptyset$	$12345\overline{134}$	$12\overline{34}$
135	24	$\emptyset$	$12345\overline{1}35$	$12\overline{35}$
234	5	1	$2345\overline{1234}$	$2\overline{234}$
235	4	1	$2345\overline{1235}$	$2\overline{235}$
245	3	1	$2345\overline{1245}$	$2\overline{245}$
345	$\emptyset$	12	$345\overline{12345}$	$\overline{2345}$

- $3, 4, 5$ and $\overline{1}$ are cone vertices.
- $\mathcal{C} = \{123, 145, 2345\}$ thus the minimal non-faces of $Act_{<}(M)$ are $1\overline{23}$, $1\overline{45}$ and $2\overline{345}$.
- Embed $IN(M)$: send $1 \to 1$ and $i \to \overline{i}$ for other i.

Question

Question

How complicated is $Act_<(M)$ as a topological space? It is clearly a cone, thus contractible, but what if we remove the cone points.

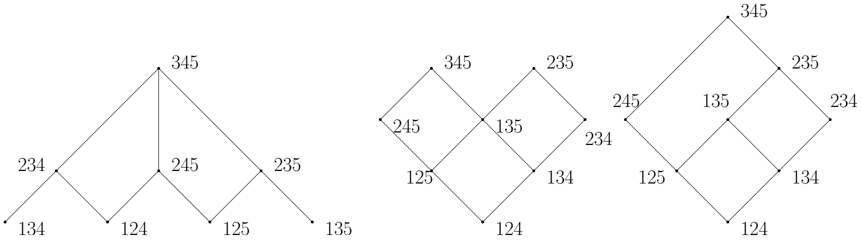
Notice that the Stanley-Reisner ring admits a natural bigrading. As a bigraded ring it understands the external activities and the broken circuit complex without even if we forget the order.

Las Vergnas active orders

We aim to study shellability and will do so in terms of linear extensions of some niece posets on $\mathcal{B}$.

- $<_{ext}$: $B <_{ext} B'$ if $B \cup EA(B) \subseteq B' \cup EA(B')$.
- $<_{int}$: dual order to $<_{ext}$ for M^*, or $B < B'$ if $IP(B) \subseteq IP(B')$.
- $<_{ext/int}$: the two orders are compatible (Las Vergnas) and $B <_{ext/int} B'$ if $B <_{ext} B'$ or $B <_{int} B'$.

Example



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Main theorem

Theorem (Ardila, Castillo, S. 14)

Every linear extension of $<_{ext/int}$ is a shelling order of $Act_<(M)$. Furthermore, we have that:

- $Act_<(M)$ is a cone over a contractible complex, unless $IN(M)$ is trivial (a cone over a join of empty simplices).
- There is an embedding of $IN(M)$ to the reduction of $Act_<(M)$ such that $IP(B) \mapsto \overline{IP(B)}$ and this enumerates the h-vector.
- This is tight in terms of the Las Vergnas orders.

Furthermore,

Theorem (Ardila, Castillo, S. 14)

Every linear extension of $<_{int}$ is a shelling order of $IN(M)$.

Questions

- Is there a way to capture all the activity in one simplicial complex. I.e is there a simplicial complex that encodes $<$ in its face structure? (It would probably be related to the Tutte Polynomial).

- $<_{int}$ is a subposet of the face poset that takes the minimal elements of the partition coming from shelling by ordering facets in lex. The fact that every linear extension of $<_{int}$ is a shelling order is a very special thing! What are the properties of shellable complexes satisfying that?

The End

Thank you very much for your attention.