A non-partitionable Cohen-Macaulay simplicial complex

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Overview: The Partitionability Conjecture

The focus of this talk is the following conjecture, described in Stanley's Green Book as "a central combinatorial conjecture on Cohen-Macaulay complexes."

Partitionability Conjecture (Stanley 1979)

Every Cohen-Macaulay simplicial complex is partitionable.

Theorem (DGKM '15+)

The Partitionability Conjecture is **false***. We construct an explicit counterexample and describe a general method to construct more.*

Partitionability

X^d = pure simplicial complex of dimension d; facets $F_1, \ldots, F_n$

A partitioning of X is a decomposition

$$X = \coprod_{j=1}^{n} [R_j,\ F_j] \qquad \text{where} \quad [R, F] \stackrel{\text{def}}{=} \{\sigma \mid R \subseteq \sigma \subseteq F\}.$$

If X is partitionable, then its h-vector has the combinatorial interpretation

$$h_i(X) = \#\{i \mid \#R_i = j\}.$$

In particular, X partitionable $\implies h(X) \geq 0$.

Partitionability and Shellability

- Every shelling order $F_1, \ldots, F_n$ gives rise to a partitioning.
- Cohen-Macaulay complexes are an important class of simplicial complexes with the same h-vectors as shellable complexes.

$$\boxed{\text{shellable} \implies \text{constructible} \implies \text{Cohen-Macaulay.}}$$

- The Partitionability Conjecture would have provided a combinatorial interpretation for the h-vectors of all Cohen-Macaulay complexes.
- Note: Our counterexample is constructible.

Algebraic Consequence: Stanley's Depth Conjecture

$$\begin{aligned} X \text{ is CM} &\iff \text{Stanley-Reisner ring } \Bbbk[X] \text{ is CM} \\ &\iff \dim \Bbbk[X] = \operatorname{depth} \Bbbk[X]. \end{aligned}$$

Stanley depth (sdepth) is an analogous combinatorial invariant.

Depth Conjecture (Stanley 1982)

Let $S = \Bbbk[x_1, \ldots, x_n]$ and $I \subset S$ be any monomial ideal. Then

$$\operatorname{sdepth} R \geq \operatorname{depth} R.$$

Theorem (Herzog–Jahan–Yassemi 2008)

The Depth Conjecture implies the Partitionability Conjecture.

Therefore, our construction disproves the Depth Conjecture.

Relative Simplicial Complexes

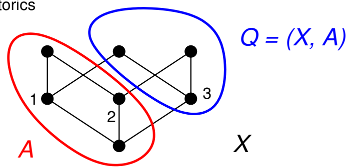
Definition

A relative simplicial complex Q on vertex set $[n]$ is a convex subset of the Boolean algebra $2^{[n]}$. That is,

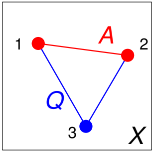
$$\sigma, \tau \in Q, \ \sigma \subseteq \rho \subseteq \tau \quad \Longrightarrow \quad \rho \in Q.$$

Every relative complex can be written as $(X, A) = X \setminus A$, where $A \subseteq X$ are simplicial complexes.

Combinatorics



Geometry



Reducing to the Relative Case

X = CM complex
$Q = (X, A)$: CM

$A \subset X$: induced, CM, codim 0 or 1
$N > \#$ faces of A

Construct Ω by gluing N copies of X together along A.

- Ω is CM by Mayer-Vietoris. On the level of face posets,

$$\Omega = Q_1 \cup \cdots \cup Q_N \cup A, \qquad Q_i \cong Q \quad \forall i.$$

- If Ω has a partitioning $\mathcal{P}$, then by pigeonhole

$$\exists\, \boldsymbol{Q_i} : \quad [\boldsymbol{R}, \boldsymbol{F}] \in \mathcal{P}, \quad \boldsymbol{F} \in \boldsymbol{Q_i} \quad \Longrightarrow \quad \boldsymbol{R} \notin \boldsymbol{A}.$$

- Therefore, the partitioning of Ω induces a partitioning of Q.

Problem: Find a suitable Q.

Background: Unshellable Balls

Mary Ellen Rudin (1958) constructed a simplicial 3-ball that is not shellable, with f-vector $(1, 14, 66, 94, 41)$ and h-vector $(1, 10, 30)$.

Günter Ziegler (1998) constructed a smaller non-shellable simplicial 3-ball **Z**, with f-vector $(1, 10, 38, 50, 21)$ and h-vector $(1, 6, 14)$. Its facets are

0123	0125	0237	0256	0267	1234	1249
1256	1269	1347	1457	1458	1489	1569
1589	2348	2367	2368	3478	3678	4578

Our Counterexample

Theorem (DGKM 2015+)

Let Z be Ziegler's ball, and let $B = Z|_{0,2,3,4,6,7,8}$.

1. *B is a shellable, hence CM, simplicial 3-ball. It $f(B) = (1, 7, 18, 19, 7)$. So it is CM (in fact it is shellable).*
2. *$Q = (Z, B)$ is relative CM, and not partitionable. Its minimal faces are the three vertices 1, 5, 9.*
3. *Therefore, the simplicial complex obtained by gluing $(1 + 7 + 18 + 19 + 7) + 1 = 53$ copies of Z together along B is a counterexample to the Partitionability Conjecture.*

Assertion (2) can be proved by elementary methods.

A Smaller Counterexample

- The complex $Q = (Z, B)$ can be expressed most efficiently as a relative complex (X, A) with

$$f(X) = (1, 10, 31, 36, 14), \qquad f(A) = (1, 7, 11, 5).$$

- So a much smaller counterexample can be constructed by gluing together $(1 + 7 + 11 + 5) + 1 = 25$ copies of Z along A.
- In fact, gluing three copies of X along A produces a counterexample Ω, with

$$\boldsymbol{f(\Omega) = 3f(X) - 2f(A) = (1, 16, 71, 98, 42).}$$

- This is the smallest counterexample we know.

Some Open Questions

- Is there a smaller counterexample, perhaps in dimension 2?
- What is the "right" strengthening of constructibility that implies partitionability? ("Strongly constructible" complexes, as studied by Hachimori, are partitionable.)
- Is there a different combinatorial interpretation of the h-vectors of Cohen-Macaulay complexes? (Yes; it's coming.)
- Are all simplicial balls partitionable? (Yes if they have a convex embedding.)
- What are the further consequences for the theory of Stanley depth? (Katthän conjectures that sdepth $R \geq$ depth $R - 1$.)

Garsia's Conjecture

Conjecture (Garsia 1979)

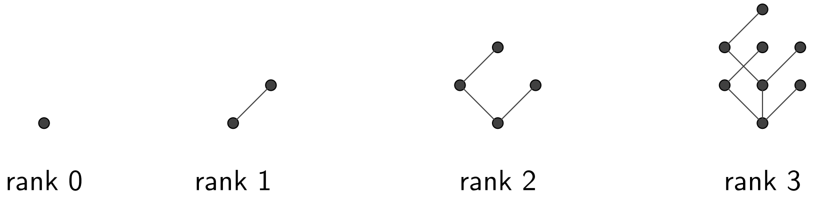
Let P be a Cohen-Macaulay poset (i.e., a ranked poset whose order complex $\Delta(P)$ is Cohen-Macaulay). Then $\Delta(P)$ is partitionable.

This conjecture remains open, as our counterexample is not even balanced, let alone an order complex.

Duval and Zhang's Interpretation of $h(\Delta)$

Theorem (Duval–Zhang 2001)

If Δ is Cohen-Macaulay, then its face poset admits a decomposition into Boolean trees whose tops are facets and whose bottoms are enumerated by $h(\Delta)$.



A Colorful Duval-Zhang-type Conjecture

Recall that a (pure) simplicial complex Δ^{d-1} is balanced if its vertex set can be colored with d colors so that no face contains more than one vertex of any color.

The flag f-vector of a balanced complex has entries

$$\alpha_S(\Delta) = \#\{\sigma \in \Delta \mid \text{color}(\sigma) = S\}, \qquad S \subseteq [d]$$

and the flag h-vector has entries

$$\beta_S(\Delta) = \sum_{T \subseteq S} (-1)^{|S|-|T|} \alpha_S(\Delta).$$

If Δ is balanced and CM then the flag h-vector is nonnegative.

A Colorful Duval-Zhang-type Conjecture

Conjecture

If Δ is balanced and Cohen-Macaulay, then its face poset admits a decomposition into balanced Boolean trees whose tops are facets and whose bottoms are enumerated by the flag h-vector.

Goeckner is currently working on extending the Duval-Zhang argument to prove this conjecture.

Thanks for listening!

Appendix A: Stanley Depth

Definition

Let $S = \mathbb{k}[x_1, \ldots, x_n]$; $\mu \in S$ a monomial; and $X \subseteq \{x_1, \ldots, x_n\}$. The corresponding Stanley space in S is the vector space

$$\mu \cdot \mathbb{k}[X] \;=\; \mathbb{k}\text{-}\operatorname{span}\{\mu\nu \mid \operatorname{supp}(\nu) \subseteq X\}.$$

Let $I \subseteq S$ be a monomial ideal. A Stanley decomposition of S/I is a family of Stanley spaces

$$\mathcal{D} \;=\; \{\mu_1 \cdot \mathbb{k}[X_1],\; \ldots,\; \mu_r \cdot \mathbb{k}[X_r]\}$$

such that

$$S/I \;=\; \bigoplus_{i=1}^{r} \mu_i \cdot \mathbb{k}[X_i].$$

Appendix A: Stanley Depth

Definition

The Stanley depth of S/I is

$$\operatorname{sdepth} S/I \;=\; \max_{\mathcal{D}} \min\{|X_i|\}.$$

where $\mathcal{D}$ runs over all Stanley decompositions of S/I.

For a nice introduction, see M. Pournaki, S. Fakhari, M. Tousi and S. Yassemi, "What is Stanley depth?", Notices AMS 2009

Appendix B: A Small Relative Counterexample

There is a much smaller relative counterexample to the Partitionability Conjecture inside Ziegler's ball Z.

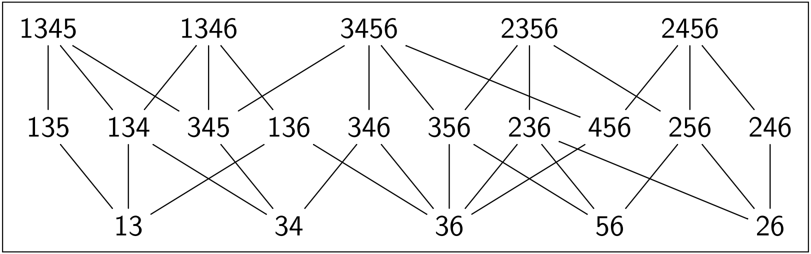
It is $Q' = (X', A')$, where

$$X' = \langle 1589,\ 1489,\ 1458,\ 1457,\ 4578 \rangle \ = Z|_{145789},$$
$$A' = \langle 489,\ 589,\ 578,\ 157 \rangle.$$

- Q' is CM (since X', A' are shellable and $A' \subset \partial X'$)
- $f(Q') = (0, 0, 5, 10, 5)$.
- Minimal faces are edges rather than vertices, so Q' cannot be expressed as (X, A) where A is an *induced* subcomplex.

Appendix B: A Small Relative Counterexample

Here's the face poset of Q':



A partitioning of Q' would correspond to a decomposition of this poset into five pairwise-disjoint diamonds.

It is not hard to check by hand that no such decomposition exists.