Toric heaps and cyclic reducibility in Coxeter groups

Matthew Macauley

Department of Mathematical Sciences
Clemson University
Clemson, SC

AMS Sectional
University of Memphis
Memphis, TN
October 17, 2015
Outline

1. Heaps & Reducibility in Coxeter groups
   - Coxeter groups
   - Heaps
   - Coxeter elements

2. Cyclic Reducibility & Conjugacy in Coxeter groups
   - Source-to-sink (toric) equivalence
   - What is cyclic reducibility?
   - Cyclically fully commutative (CFC) elements

3. Toric posets
   - Ordinary partially ordered sets
   - Toric partially ordered sets
   - Features of toric posets
   - Toric chains and total cyclic orders

4. Toric heaps & cyclic reducibility

5. Current and future research
Coxeter groups

Definition

A **Coxeter system** consists of a group $W$ (called a **Coxeter group**) generated by a set $S$ of involutions with presentation

$$W = \langle S \mid s^2 = 1, \ (st)^{m(s,t)} = 1 \rangle$$

where $m(s, t) \geq 2$ for $s \neq t$.

Since $s$ and $t$ are involutions, the relation $(st)^{m(s,t)} = 1$ means:

- $m(s, t) = 2 \implies st = ts$ \hspace{1cm} \text{commutation (or “short braid”) relation}
- $m(s, t) = 3 \implies stst = tst$ \hspace{1cm} \text{long braid relations}
- $m(s, t) = 4 \implies ststst = tstst$

A Coxeter system can be encoded by a **Coxeter graph** $\Gamma$:

- vertex set $S$
- edge $\{s, t\}$ for each $m(s, t) \geq 3$, labeled with $m(s, t)$.

Edges represent **non-commuting pairs** of generators.
Heaps

A heap is a labeled poset, and a convenient way to visualize words in Coxeter groups.

Example

Consider the element $w = s_1 s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1 s_2$ in the Coxeter group $W(H_3)$.

Imagine dropping labeled balls onto the Coxeter graph $\Gamma$ in a “Towers of Hanoi” fashion:

This visualization doesn’t work well for complicated graphs, but a heap abstracts this concept.
Formalizing the stack of balls

Key properties (which subsets are totally ordered)

- Any subset of balls with the same label $s$ is vertically aligned.
- Any subset of balls with adjacent labels $s$ and $t$ overlap vertically.

The heap is the poset generated by these subsets as chains (totally ordered sets).

Definition (Green, 2010)

A heap is a map $\phi: P \rightarrow \Gamma$ satisfying:

- For every vertex $s$ and edge $\{s, t\}$ of $\Gamma$, the subsets $\phi^{-1}(\{s\})$ and $\phi^{-1}(\{s, t\})$ are chains in $P$.
- If $P'$ is another partial order on the same set containing these chains, then $P'$ is an extension of $P$. 
A special case: The heap of a Coxeter element is an acyclic orientation of $\Gamma$.

Start with $w = s_1 s_3 s_2 s_4 = s_3 s_1 s_2 s_4 = s_3 s_1 s_4 s_2 = s_1 s_3 s_4 s_2 \in W(\tilde{A}_3)$.

Conjugate by $s_3$: $s_3(s_3 s_1 s_2 s_4) s_3 = s_1 s_2 s_4 s_3 = s_1 s_4 s_2 s_3$.

Conjugate by $s_1$: $s_1(s_1 s_2 s_4 s_3) s_1 = s_2 s_4 s_3 s_1 = s_4 s_2 s_3 s_1$
$= s_2 s_4 s_1 s_3 = s_4 s_2 s_1 s_3$.

Conjugate by $s_4$: $s_4(s_4 s_2 s_3 s_1) s_4 = s_2 s_3 s_1 s_4 = s_2 s_1 s_3 s_4$.

Conjugate by $s_2$: $s_2(s_2 s_1 s_3 s_4) s_2 = s_1 s_3 s_4 s_2 = s_3 s_1 s_4 s_2$
$= s_1 s_3 s_2 s_4 = s_3 s_1 s_2 s_4$. 
Motivation for cyclic reducibility

3 ways to describe cyclically shifting Coxeter elements

- conjugating by an initial generator: \( s_{x_1} (s_{x_1} s_{x_2} \cdots s_{x_n}) s_{x_1} = s_{x_2} \cdots s_{x_n} s_{x_1} \) (in \( W \)).
- making a minimal element maximal (in the heap)
- converting a source vertex into a sink (in the acyclic orientation).

This defines toric equivalence, \( \equiv \) on \( \text{Acyc}(\Gamma) \) and on \( C(W) \).

Example (revisited)

Consider the Coxeter element \( c = s_1 s_3 s_2 s_4 \) in \( W(\tilde{A}_3) \).

We can think of \( c \) as being a “cyclic word” \([c]\), subject to the same relations:
Recent results on Coxeter elements

Definition

An element \( w \in W \) is logarithmic if \( \ell(w^k) = k \cdot \ell(w) \) for all \( k \in \mathbb{N} \).

An element \( w \in W \) is torsion-free if every connected component of the induced graph \( \Gamma_{\text{supp}(w)} \) describes an infinite Coxeter group.

Theorem (Speyer, 2009)

Coxeter elements are logarithmic iff they are torsion-free.

Theorem (Eriksson–Eriksson, 2009)

Two Coxeter elements \( c, c' \in W \) are conjugate iff \( c \equiv c' \).

Central to the proofs of these theorems is toric equivalence.

Open-ended question

How can these results be extended from Coxeter elements to general heaps?
Theorem (Eriksson–Eriksson, 2009)

Two Coxeter elements \( c, c' \in W \) are conjugate iff they are torically equivalent.

Thus, there are bijections between:

- \( C(W) \) (Coxeter elements) and \( \text{Acyc}(\Gamma) \) (acyclic orientations);
- \( C(W)/\equiv \) (conjugacy classes) and \( \text{Acyc}(\Gamma)/\equiv \) (toric equivalence classes).

Example

Returning to our previous example of \( W(\widetilde{A}_3) \), there are:

- \( 4! = 24 \) reduced words for a Coxeter element in \( W \);
- \( |\text{Acyc}(C_4)| = T_{C_4}(2, 0) = 14 \) distinct Coxeter elements in \( W \);
- \( |\text{Acyc}(C_4)/\equiv| = T_{C_4}(1, 0) = 3 \) distinct conjugacy classes that contain these elements:
Goals

1. Develop and study a theory of cyclic reducibility in Coxeter groups.
2. Formalize a cyclic heap and study this object.
3. Apply these cyclic heaps to Coxeter theory.

Remarks

- We will extend the ideas on the previous slide from Coxeter elements to general cyclically reduced elements.
- We will formalize “cyclic versions” of classical Coxeter concepts such as words, reduced words, braid relations, commutativity classes, and fully commutative elements.
- This “cyclic heap” will be a labeled toric poset, which is a cyclic analogue of an ordinary poset. We call this object a toric heap.
- Classical problems on reducibility become new problems in cyclic reducibility and conjugacy.
- Toric heaps are interesting to study on their own right.
Cyclically fully commutative elements

Definition (CFC elements)
An element \( w \in W \) is cyclically fully commutative (CFC) if every cyclic shift of a reduced word for \( w \) is FC.

FC vs. CFC
- Ordinary reducibility: \( w \) is FC \( \iff \) \( w \) has a unique commutativity class.
- Cyclic reducibility: \( w \) is CFC \( \implies \) \( w \) has a unique cyclic commutativity class.

Non-example
Let \( W = B_3 \), and consider the word \( w = s_1 s_2 s_1 s_2 s_3 \).

Note that \( w \) is not CFC but has a unique cyclic commutativity class. We say that such elements are faux CFC.
Cyclically fully commutative (CFC) elements

Examples in affine Weyl groups

- $\widetilde{C}_4$
  
  $u = s_0 s_1 s_2 s_3 s_4 s_3 s_2 s_1$

- $\widetilde{E}_6$
  
  $w = s_1 s_3 s_2 s_4 s_3 s_5 s_4 s_6 s_0 s_3 s_2 s_6$

Non-example

Consider the following non-CFC element $w \in W(\widetilde{C}_2)$.

$w = s_0 s_1 s_0 s_1 s_2$

Clearly, $w$ is cyclically reduced and torsion-free, but not logarithmic:

$w^2 = (s_0 s_1 s_0 s_1 s_2)(s_0 s_1 s_0 s_1 s_2) = (s_1 s_0 s_1 s_0 s_0)(s_2 s_1 s_0 s_1 s_2) = (s_1 s_0 s_1)(s_2 s_1 s_0 s_1 s_2)$. 

Matthew Macauley (Clemson)

Toric heaps & cyclic reducibility

October 17, 2015
Partially ordered sets

3 ways to characterize posets
1. a set with a binary relation;
2. an acyclic orientation (digraph);
3. a chamber of a graphic hyperplane arrangement $\mathcal{A}(G)$ in $\mathbb{R}^n$.

Example (4 posets over $L_3$)
2 ways to characterize toric posets

1. a set with a binary relation.
2. an equivalence class of acyclic orientations (under source-to-sink operations);
3. a chamber of a graphic toric hyperplane arrangement $\mathcal{A}_{\text{tor}}(G)$ in $\mathbb{R}^n/\mathbb{Z}^n$.

Example
Features of toric posets

Many common features of ordinary posets have elegant toric analogues, such as:

- directed paths
- extensions
- total orders
- chains
- transitivity (toric transitive closure is convex!)
- Hasse diagrams
- antichains
- intervals
- morphisms
- order ideals & filters

Most of these do not become apparent until one interprets the classical definition geometrically, and then passes to the quotient $\pi : \mathbb{R}^V \rightarrow \mathbb{R}^V / \mathbb{Z}^V$:

$$\begin{array}{c}
\mathbb{R}^V - \mathcal{A}(G) \xrightarrow{\pi} \mathbb{R}^V / \mathbb{Z}^V - \mathcal{A}_{\text{tor}}(G) \\
\bar{\alpha}_G \downarrow \quad \alpha_G \\
\text{Acyc}(G) \quad ?? \quad \rightarrow \quad \text{Acyc}(G) / \equiv
\end{array}$$
Toric chains

Chains of posets

A set $C = \{i_1, \ldots, i_k\}$ is a chain of $P(G, \omega)$ if

- $C$ lies on a directed path of $\omega$;
- $C$ is totally ordered.

Toric chains of toric posets

A set $C = \{i_1, \ldots, i_k\}$ is a toric chain of $P(G, [\omega])$ if

- $C$ lies on a toric directed path of $\omega$;
- $C$ is totally cyclically ordered.

Example: $\{1, 2, 3, 4\}$ is totally cyclically ordered

![Diagram showing cyclically ordered sets]
**Toric chains**

### Chains of posets

A set \( C = \{i_1, \ldots, i_k\} \) is a chain of \( P(G, \omega) \) if

- \( C \) lies on a directed path of \( \omega \);
- \( C \) is totally ordered.

\[ i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow \cdots \rightarrow i_k \]

### Toric chains of toric posets

A set \( C = \{i_1, \ldots, i_k\} \) is a toric chain of \( P(G, [\omega]) \) if

- \( C \) lies on a toric directed path of \( \omega \);
- \( C \) is totally cyclically ordered.

\[ i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow \cdots \rightarrow i_k \]

### Non-example \( \{1, 2, 3, 4\} \) is not totally cyclically ordered

\[ 4 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow 3 \rightarrow 4 \]

\[ 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 1 \]

\[ 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 1 \]

\[ 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 1 \]

\[ 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 1 \]

\[ 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 1 \]
Toric heaps formalized

Definition

A toric heap is a map $\tau : T \rightarrow \Gamma$ from a toric poset $T$ satisfying:

- For every vertex $s$ and edge $\{s, t\}$ of $\Gamma$, the subsets $\tau^{-1}(\{s\})$ and $\tau^{-1}(\{s, t\})$ are toric chains in $T$ (totally cyclically ordered).

- If $T'$ is another toric partial order on the same set containing these toric chains, then $T'$ is an extension of $T$.

Just like how every reduced expression $w \in W$ determines a heap $P_w$, every cyclically reduced expression determines a toric heap $T_w$.

Proposition (Chao, M–; 2015)

The toric heaps $T_w$ and $T_u$ are isomorphic:

- if $u \in [w]$; (cyclic shifts)
- if $[u] \sim [w]$. (differ by commutation relations)

Corollary

An element has a unique toric heap if and only if it is CFC or faux CFC.
Heaps vs. toric heaps

**Example**

Consider \( w = s_1 s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1 s_2 \) in \( W(H_3) \).

![Coxeter graph Γ, heap of \( w \), toric heap of \( w \)]

**Remarks**

- The toric heap contains two extra edges which make both edge chains into toric edge chains. \([s_1, s_2, s_1, s_2, \text{ and } s_3, s_2, s_3 \text{ are totally cyclically ordered} \]
- These edges are needed for the source-to-sink operation to properly correspond to conjugacy.
Current and future research

- Finish the paper on toric heaps, cyclic reducibility & conjugacy in Coxeter groups.

- Can the techniques of Speyer, and of Eriksson–Eriksson be extended from Coxeter elements to CFC elements using toric heaps instead of equivalence classes of acyclic orientations?

- How can one enumerate the number of total toric extensions of a toric poset? Is there a “toric hook-length formula”?

- Explore the toric analogue of \( P \)-partitions. Can these enumerate lattice points in regions in toric graphic arrangements? Is there a toric Ehrhart reciprocity theory?

- Are there toric analogues of the Möbius function, or the order complex? If so, what information might its topology encode?

- The assembly line-balancing problem in operations research is an algorithm for scheduling a set of tasks. It assumes there is a natural poset structure on the set of tasks. If the timeline is periodic, then there should be a toric poset structure instead. Develop this theory.
References


