Toric heaps and cyclic reducibility in Coxeter groups

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Outline

1. Heaps & Reducibility in Coxeter groups
    - Coxeter groups
    - Heaps
    - Coxeter elements
2. Cyclic Reducibility & Conjugacy in Coxeter groups
    - Source-to-sink (toric) equvalence
    - What is cyclic reducibility?
    - Cyclically fully commutative (CFC) elements
3. Toric posets
    - Ordinary partially ordered sets
    - Toric partially ordered sets
    - Features of toric posets
    - Toric chains and total cyclic orders
4. Toric heaps & cyclic reducibility
5. Current and future research

Coxeter groups

Definition

A *Coxeter system* consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid \; s^2 = 1, \quad (st)^{m(s,t)} = 1 \rangle$$

where $m(s,t) \geq 2$ for $s \neq t$.

Since s and t are involutions, the relation $(st)^{m(s,t)} = 1$ means:

$$m(s,t) = 2 \quad \Longrightarrow \quad st = ts \quad \Big\} \quad \text{commutation (or ``short braid'') relation}$$

$$\left.\begin{array}{lcl} m(s,t) = 3 & \Longrightarrow & sts = tst \\ m(s,t) = 4 & \Longrightarrow & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{long braid relations}$$

A Coxeter system can be encoded by a Coxeter graph Γ:

- vertex set S
- edge $\{s,t\}$ for each $m(s,t) \geq 3$, labeled with $m(s,t)$.

Edges represent non-commuting pairs of generators.

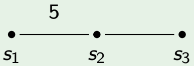
Heaps

A heap is a labeled poset, and a convenient way to visualize words in Coxeter groups.

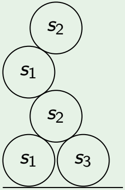
Example

Consider the element $w = s_1 s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1 s_2$ in the Coxeter group $W(H_3)$.

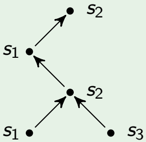
Imagine dropping labeled balls onto the Coxeter graph Γ in a "Towers of Hanoi" fashion:



Coxeter graph Γ



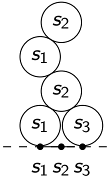
$w = s_1 s_3 s_2 s_1 s_2$

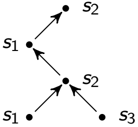


Heap (labeled poset) of w

This visualization doesn't work well for complicated graphs, but a heap abstracts this concept.

Formalizing the stack of balls





Key properties (which subsets are totally ordered)

- Any subset of balls with the same label s is vertically aligned.
- Any subset of balls with adjacent labels s and t overlap vertically.

The heap is the poset generated by these subsets as chains (totally ordered sets).

Definition (Green, 2010)

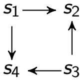
A heap is a map $\phi\colon P \longrightarrow \Gamma$ satisfying:

- For every vertex s and edge $\{s,t\}$ of Γ, the subsets $\phi^{-1}(\{s\})$ and $\phi^{-1}(\{s,t\})$ are chains in P.
- If P' is another partial order on the same set containing these chains, then P' is an extension of P.

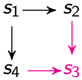
Coxeter elements & acyclic orientations

A special case: The heap of a Coxeter element is an acyclic orientation of Γ.

Start with $w = s_1 s_3 s_2 s_4 = s_3 s_1 s_2 s_4 = s_3 s_1 s_4 s_2 = s_1 s_3 s_4 s_2 \in W(\widetilde{A_3})$.

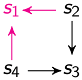


Conjugate by s_3: $\quad s_3(\boldsymbol{s_3} s_1 s_2 s_4)s_3 = s_1 s_2 s_4 \boldsymbol{s_3} = s_1 s_4 s_2 \boldsymbol{s_3}$.

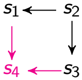


Conjugate by s_1: $\quad s_1(\boldsymbol{s_1} s_2 s_4 s_3)s_1 = s_2 s_4 s_3 \boldsymbol{s_1} = s_4 s_2 s_3 \boldsymbol{s_1}$

$= s_2 s_4 \boldsymbol{s_1} s_3 = s_4 s_2 \boldsymbol{s_1} s_3$.

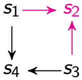


Conjugate by s_4: $\quad s_4(\boldsymbol{s_4} s_2 s_3 s_1)s_4 = s_2 s_3 s_1 \boldsymbol{s_4} = s_2 s_1 s_3 \boldsymbol{s_4}$.



Conjugate by s_2: $\quad s_2(\boldsymbol{s_2} s_1 s_3 s_4)s_2 = s_1 s_3 s_4 \boldsymbol{s_2} = s_3 s_1 s_4 \boldsymbol{s_2}$

$= s_1 s_3 \boldsymbol{s_2} s_4 = s_3 s_1 \boldsymbol{s_2} s_4$.



Motivation for cyclic reducibility

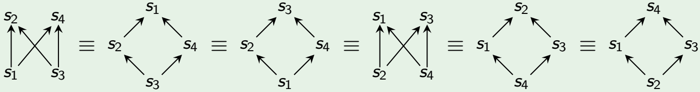
3 ways to describe cyclically shifting Coxeter elements

- conjugating by an initial generator: $s_{x_1}(s_{x_1}s_{x_2}\cdots s_{x_n})s_{x_1} = s_{x_2}\cdots s_{x_n}s_{x_1}$ (in W).
- making a minimal element maximal (in the heap)
- converting a source vertex into a sink (in the acyclic orientation).

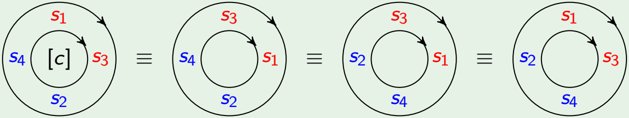
This defines toric equivalence, $\equiv$ on $\mathrm{Acyc}(\Gamma)$ and on $\mathrm{C}(W)$.

Example (revisted)

Consider the Coxeter element $c = s_1s_3s_2s_4$ in $W(\widetilde{A_3})$.



We can think of c as being a “cyclic word” $[c]$, subject to the same relations:



Recent results on Coxeter elements

Definition

An element $w \in W$ is logarithmic if $\ell(w^k) = k \cdot \ell(w)$ for all $k \in \mathbb{N}$.

An element $w \in W$ is torsion-free if every connected component of the induced graph $\Gamma_{\mathrm{supp}(w)}$ describes an infinite Coxeter group.

Theorem (Speyer, 2009)

Coxeter elements are logarithmic iff they are torsion-free.

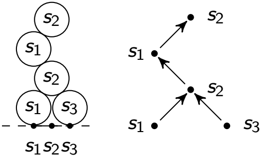
Theorem (Eriksson–Eriksson, 2009)

Two Coxeter elements $c, c' \in W$ are conjugate iff $c \equiv c'$.

Central to the proofs of these theorems is toric equivalence.

Open-ended question

How can these results be extended from Coxeter elements to general heaps?



Conjugation in Coxeter groups

Theorem (Eriksson–Eriksson, 2009)

Two Coxeter elements $c, c' \in W$ are conjugate iff they are torically equivalent.

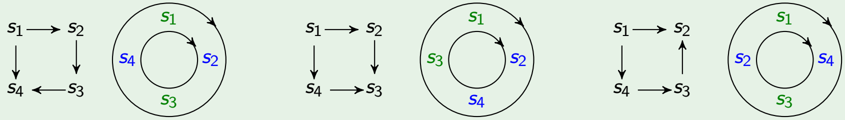
Thus, there are bijections between:

- $\mathrm{C}(W)$ (Coxeter elements) and $\mathrm{Acyc}(\Gamma)$ (acyclic orientations);
- $\mathrm{C}(W)/\!\equiv$ (conjugacy classes) and $\mathrm{Acyc}(\Gamma)/\!\equiv$ (toric equivalence classes).

Example

Returning to our previous example of $W(\widetilde{A_3})$, there are:

- $4! = 24$ reduced words for a Coxeter element in W;
- $|\mathrm{Acyc}(C_4)| = T_{C_4}(2,0) = 14$ distinct Coxeter elements in W;
- $|\mathrm{Acyc}(C_4)/\!\equiv| = T_{C_4}(1,0) = 3$ distinct conjugacy classes that contain these elements:



Talk overview in one slide

Goals

1. Develop and study a theory of cyclic reducibility in Coxeter groups.
2. Formalize a cyclic heap and study this object.
3. Apply these cyclic heaps to Coxeter theory.

Remarks

- We will extend the ideas on the previous slide from Coxeter elements to general cyclically reduced elements.
- We will formalize "cyclic versions" of classical Coxeter concepts such as words, reduced words, braid relations, commutativity classes, and fully commutative elements.
- This "cyclic heap" will be a labeled toric poset, which is a cyclic analogue of an ordinary poset. We call this object a toric heap.
- Classical problems on reducibility become new problems in cyclic reducibility and conjugacy.
- Toric heaps are interesting to study on their own right.

Cyclically fully commutative elements

Definition (CFC elements)

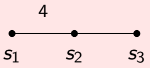
An element $w \in W$ is cyclically fully commutative (CFC) if every cyclic shift of a reduced word for w is FC.

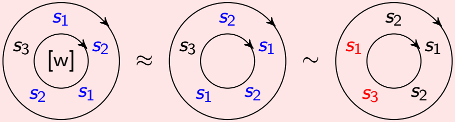
FC vs. CFC

- Ordinary reducibility: w is FC $\Longleftrightarrow$ w has a unique commutativity class.
- Cyclic reducibility: w is CFC $\Longrightarrow$ w has a unique cyclic commutativity class.

Non-example

Let $W = B_3$, and consider the word $\mathrm{w} = s_1 s_2 s_1 s_2 s_3$.



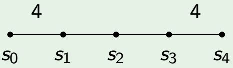


Note that w is not CFC but has a unique cyclic commutativity class. We say that such elements are faux CFC.

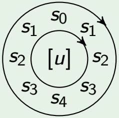
Cyclically fully commutative (CFC) elements

Examples in affine Weyl groups

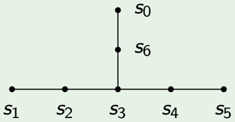
- $\widetilde{C}_4$



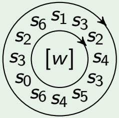
$u = s_0 s_1 s_2 s_3 s_4 s_3 s_2 s_1$



- $\widetilde{E}_6$



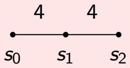
$w = s_1 s_3 s_2 s_4 s_3 s_5 s_4 s_6 s_0 s_3 s_2 s_6$



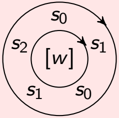
Non-example

Consider the following non-CFC element $w \in W(\widetilde{C}_2)$.

$\widetilde{C}_2$



$w = s_0 s_1 s_0 s_1 s_2$



Clearly, w is cyclically reduced and torsion-free, but *not* logarithmic:

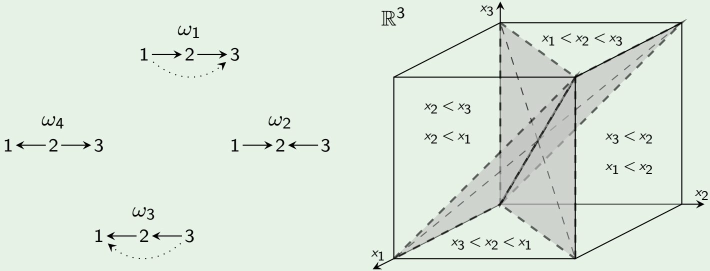
$$w^2 = (s_0 s_1 s_0 s_1 s_2)(s_0 s_1 s_0 s_1 s_2) = (s_1 s_0 s_1 s_0 s_0)(s_2 s_1 s_0 s_1 s_2) = (s_1 s_0 s_1)(s_2 s_1 s_0 s_1 s_2)\,.$$

Partially ordered sets

3 ways to characterize posets

1. a set with a binary relation;
2. an acyclic orientation (digraph);
3. a chamber of a graphic hyperplane arrangement $\mathcal{A}(G)$ in $\mathbb{R}^n$.

Example (4 posets over L_3)

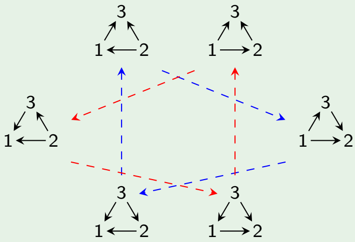


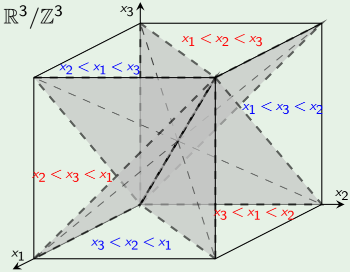
Toric partially order sets

2 ways to characterize toric posets

1. ~~a set with a binary relation.~~
2. an equivalence class of acyclic orientations (under source-to-sink operations);
3. a chamber of a graphic toric hyperplane arrangment $\mathcal{A}_{\text{tor}}(G)$ in $\mathbb{R}^n/\mathbb{Z}^n$.

Example





Features of toric posets

Many common features of ordinary posets have elegant toric analogues, such as:

- directed paths
- extensions
- total orders
- chains
- transitivity (toric transitive closure is convex!)
- Hasse diagrams
- antichains
- intervals
- morphisms
- order ideals & filters

Most of these do not become apparent until one interprets the classical definition geometrically, and then passes to the quotient $\pi\colon \mathbb{R}^V \longrightarrow \mathbb{R}^V/\mathbb{Z}^V$:

$$\begin{tikzcd}
\mathbb{R}^V - \mathcal{A}(G) \arrow[r, "\pi"] \arrow[d, "\alpha_G"'] & \mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\text{tor}}(G) \arrow[d, "\bar{\alpha}_G"] \\
\text{Acyc}(G) \arrow[r, dotted, "???"] & \text{Acyc}(G)/\equiv
\end{tikzcd}$$

Toric chains

Chains of posets

A set $C = \{i_1, \ldots, i_k\}$ is a chain of $P(G, \omega)$ if

- C lies on a directed path of ω;
- C is totally ordered.

$$\begin{tikzcd} i_1 \arrow[r] & i_2 \arrow[r] & i_3 \arrow[r] & \cdots \arrow[r] & i_k \end{tikzcd}$$

Toric chains of toric posets

A set $C = \{i_1, \ldots, i_k\}$ is a toric chain of $P(G, [\omega])$ if

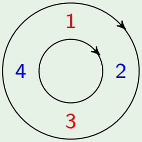
- C lies on a *toric directed path* of ω;
- C is totally cyclically ordered.

$$\begin{tikzcd} i_1 \arrow[r] \arrow[rrrr, bend left] & i_2 \arrow[r] & i_3 \arrow[r] & \cdots \arrow[r] & i_k \end{tikzcd}$$

Example: $\{1, 2, 3, 4\}$ *is* totally cyclically ordered

$$\begin{tikzcd} 1 \arrow[r] \arrow[rrr, bend left] & 2 \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[r] \arrow[rrr, bend left] & 3 \arrow[r] & 4 \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} 3 \arrow[r] \arrow[rrr, bend left] & 4 \arrow[r] & 1 \arrow[r] & 2 \end{tikzcd} \qquad \begin{tikzcd} 4 \arrow[r] \arrow[rrr, bend left] & 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd}$$



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Toric chains of toric posets

A set $C = \{i_1, \dots, i_k\}$ is a toric chain of $P(G, [\omega])$ if

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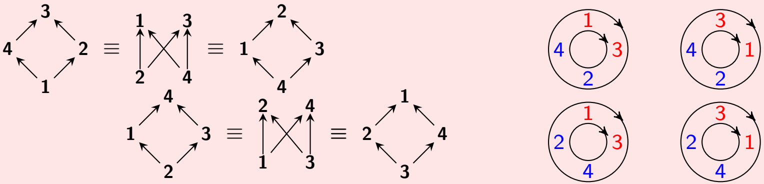
$$\begin{tikzcd} i_1 \arrow[r] \arrow[rrrr, bend left] & i_2 \arrow[r] & i_3 \arrow[r] & \cdots \arrow[r] & i_k \end{tikzcd}$$

Non-example $\{1, 2, 3, 4\}$ is *not* totally cyclically ordered



Toric heaps formalized

Definition

A toric heap is a map $\tau\colon T \longrightarrow \Gamma$ from a toric poset T satisfying:

- For every vertex s and edge $\{s, t\}$ of Γ, the subsets $\tau^{-1}(\{s\})$ and $\tau^{-1}(\{s, t\})$ are toric chains in T (totally cyclically ordered).
- If T' is another toric partial order on the same set containing these toric chains, then T' is an extension of T.

Just like how every reduced expression $w \in W$ determines a heap P_{w}, every cyclically reduced expression determines a toric heap T_{w}.

Proposition (Chao, M–; 2015)

The toric heaps T_{w} and T_{u} are isomorphic:

- if $\mathrm{u} \in [\mathrm{w}]$; (cyclic shifts)
- if $[\mathrm{u}] \sim [\mathrm{w}]$. (differ by commutation relations)

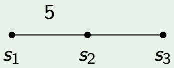
Corollary

An element has a unique toric heap if and only if it is CFC or faux CFC.

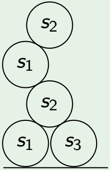
Heaps vs. toric heaps

Example

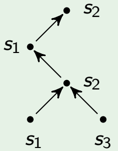
Consider $w = s_1 s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1 s_2$ in $W(H_3)$.



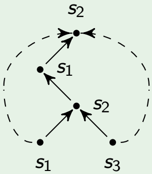
Coxeter graph Γ



$w = s_1 s_3 s_2 s_1 s_2$



Heap of w
(labeled poset)



Toric heap of w
(labeled toric poset)

Remarks

- The toric heap contains two extra edges which make both edge chains into toric edge chains. [s_1, s_2, s_1, s_2, and s_3, s_2, s_3 are totally cyclically ordered]
- These edges are needed for the source-to-sink operation to properly correspond to conjugacy.

Current and future research

- Finish the paper on toric heaps, cyclic reducibility & conjugacy in Coxeter groups.
- Can the techniques of Speyer, and of Eriksson–Eriksson be extended from Coxeter elements to CFC elements using toric heaps instead of equivalence classes of acyclic orientations?
- How can one enumerate the number of total toric extensions of a toric poset? Is there a "toric hook-length formula"?
- Explore the toric analogue of P-partitions. Can these enumerate lattice points in regions in toric graphic arrangements? Is there a toric Ehrhart reciprocity theory?
- Are there toric analogues of the Möbius function, or the order complex? If so, what information might its topology encode?
- The assembly line-balancing problem in operations research is an algorithm for scheduling a set of tasks. It assumes there is a natural poset structure on the set of tasks. If the timeline is periodic, then there should be a toric poset structure instead. Develop this theory.

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