Simplicial Moves on Balanced Manifolds

Steven Klee

joint with Ivan Izmestiev (U. Fribourg) and Isabella Novik (U. Washington)

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Special Session on Topological Combinatorics

University of Memphis

October 17, 2015

Definition

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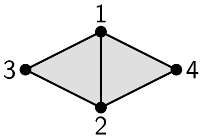
Δ = pure d-dimensional simplicial complex
$A \in \Delta$
$\text{lk}_\Delta(A) = \partial B$ ($B \notin \Delta$ and $\dim B = d - \dim A$)
The exchange $A * \partial B \to \partial A * B$ is a *bistellar flip* on Δ.



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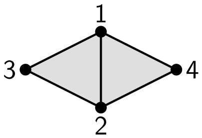
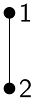
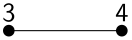


$A =$ 

$B =$ 

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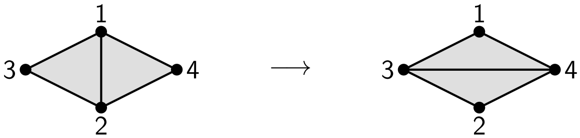

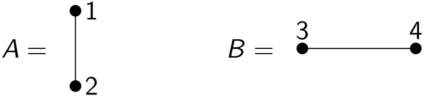

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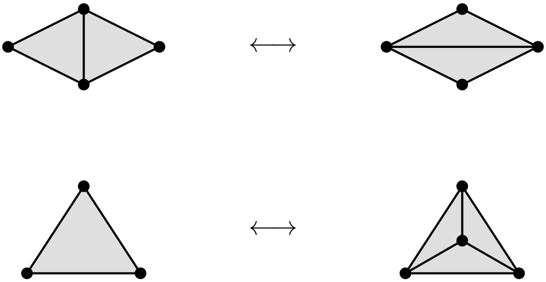

Motivation

Theorem (Pachner)

Δ, Δ' closed PL manifolds

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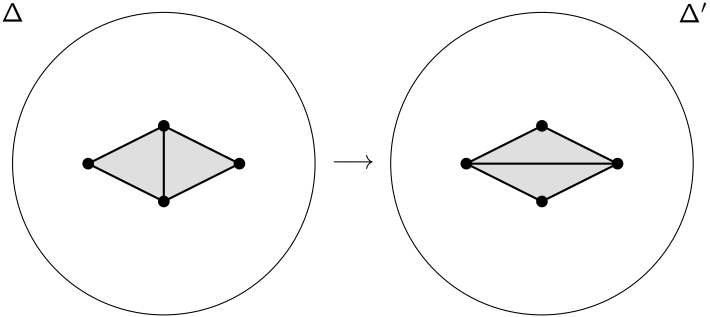
- On a d-manifold, there are $d+2$ possible bistellar flips.
- Lutz's BISTELLAR package allows computational testing *and* searching.

Viewing bistellar flips geometrically



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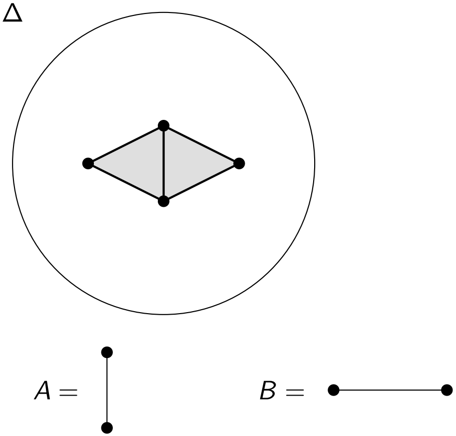

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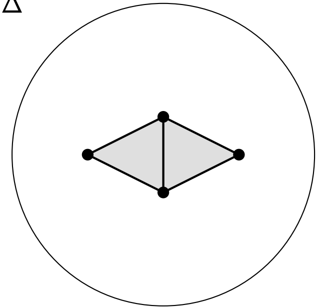
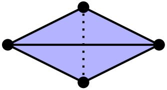


$A * B$



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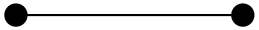

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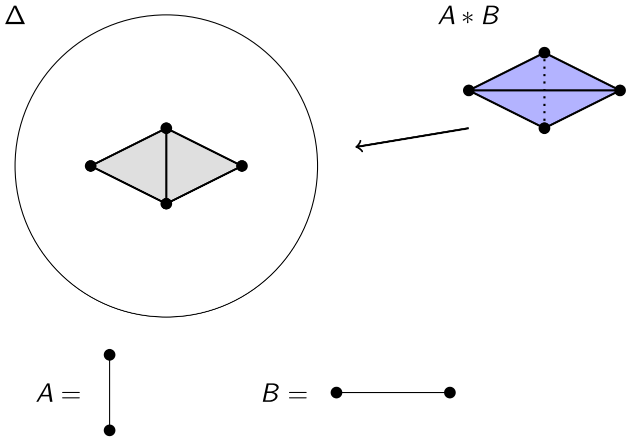

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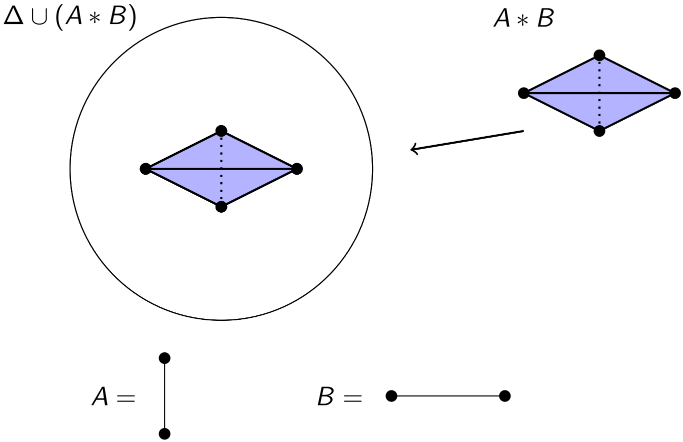

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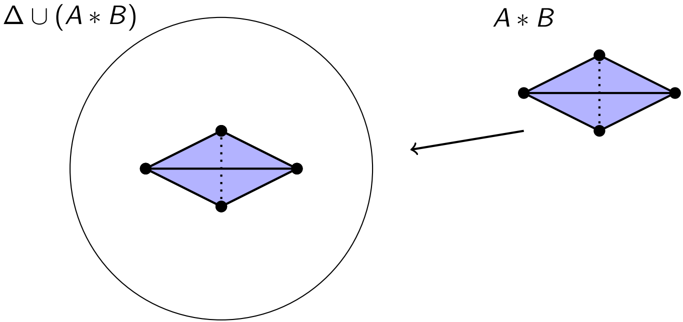
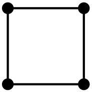


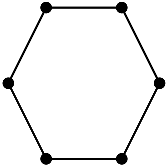
Δ = view of $\Delta \cup (A * B)$ "from below."
Δ' = view of $\Delta \cup (A * B)$ "from above."

Another example

Δ



Δ'



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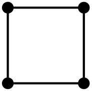

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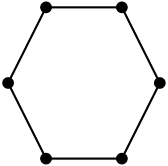

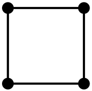

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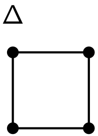

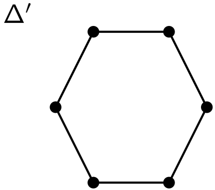

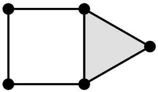

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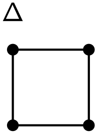

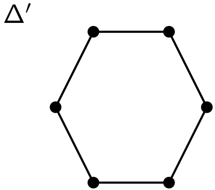

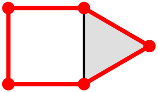

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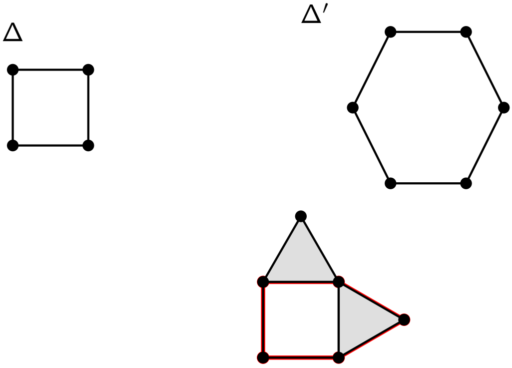

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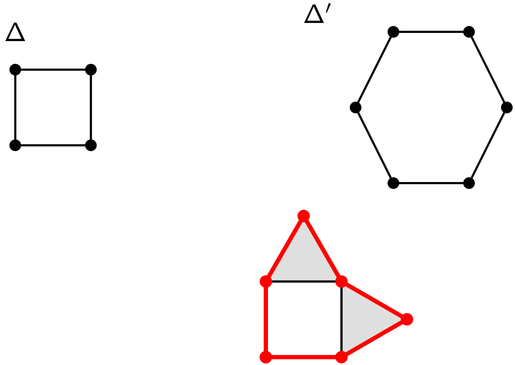

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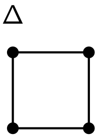

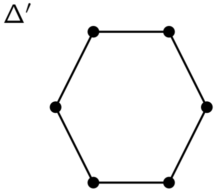

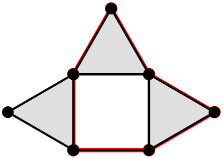

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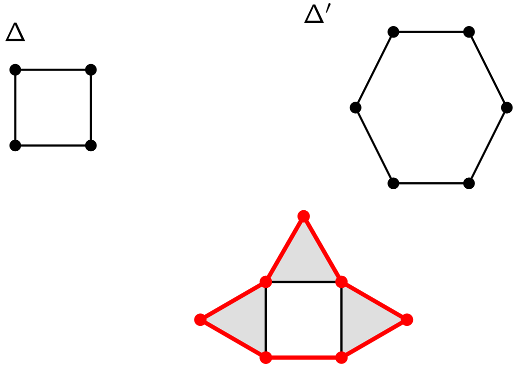

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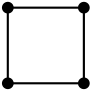

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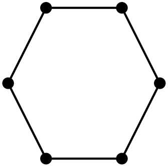

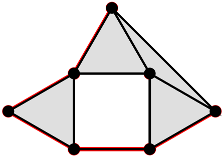

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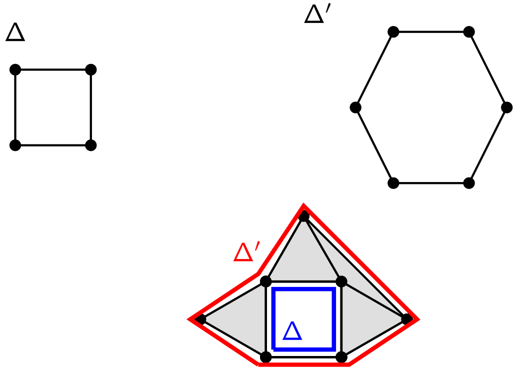

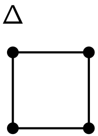
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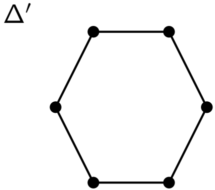
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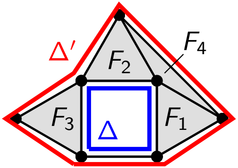
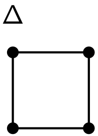
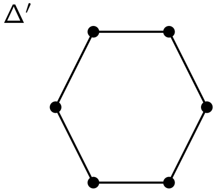
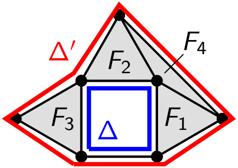

Δ′
F2
F4
F3
Δ
F1

Another example





Ω :



Ω is a *shellable pseudo-cobordism* between Δ and Δ'

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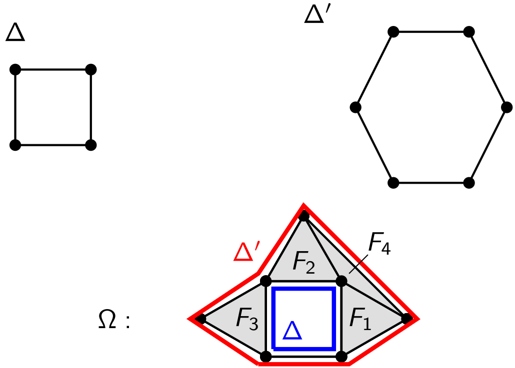

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F_1, F_2, F_3, F_4 is a shelling order on (Ω, Δ)

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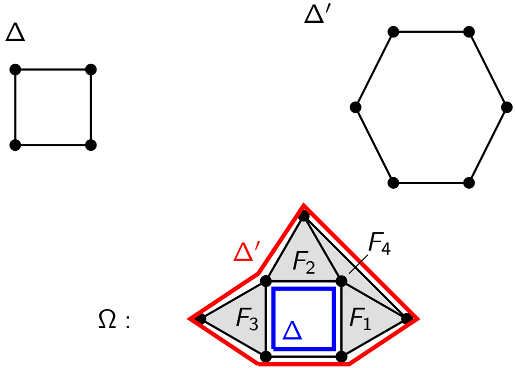

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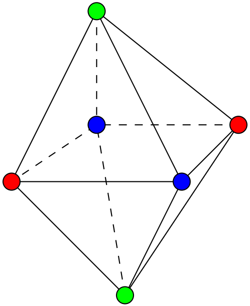
Balanced simplicial complexes

Definition (Stanley)

Δ = d-dimensional simplicial complex
Δ is **balanced** if its graph is properly $(d+1)$-colorable



$\mathcal{C}_d^* :=$ boundary complex of a $(d+1)$-dimensional cross-polytope

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The process of replacing $X \to Y$ in Δ is called a **cross-flip** on Δ.

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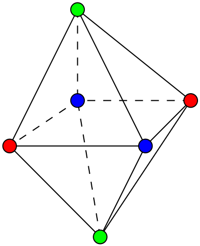

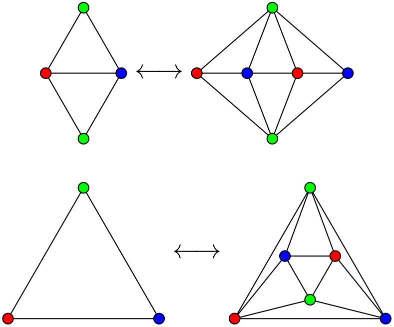

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Theorem (Izmestiev-K.-Novik '15+)

Let Δ and Δ' be balanced combinatorial manifolds.
Then Δ and Δ' are PL homeomorphic if and only if they can be connected through a sequence of cross-flips.

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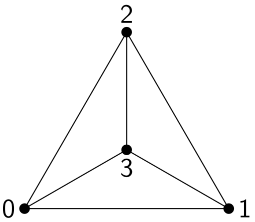

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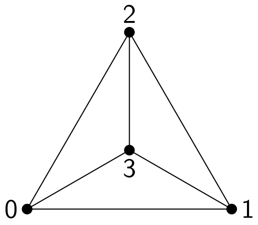

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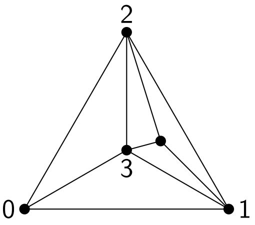

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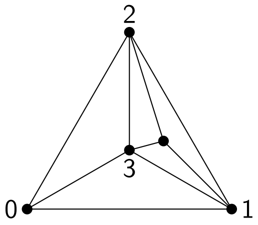

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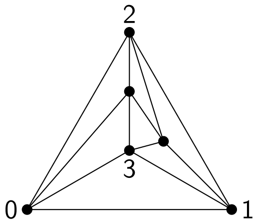

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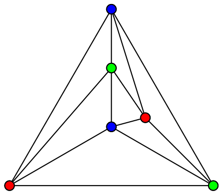

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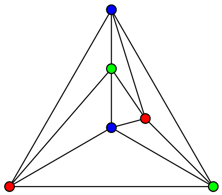
- Subdivide some more to draw cross-polytopes over the $(d+1)$-faces.



1. Avoid 0
2. Avoid $0, 1$

- Bistellar flips become cross-flips

Thank you!