On filters of the partition lattice

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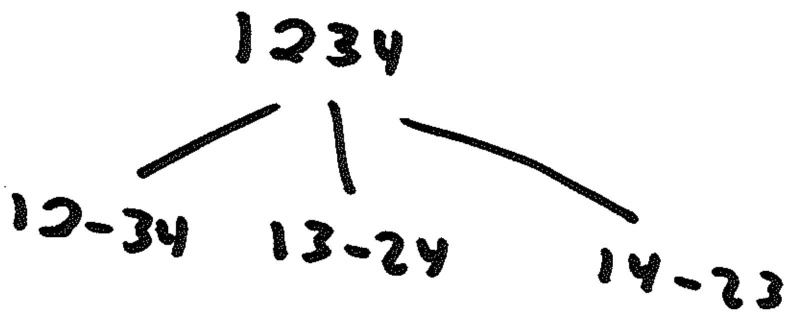
Π_n^2 = the even partition lattice
$= \{\pi \in \Pi_n : \forall B \in \pi \ \ |B| \text{ is even}\}$

$\Pi_4^2 =$



Theorem [Sylvester] $\mu(\Pi_n^2 \cup \{\hat{0}\}) = (-1)^{n/2} \cdot E_{n-1}$

E_n = nth Euler number = # $\sigma \in \mathfrak{S}_n$ $\ \sigma_1 < \sigma_2 > \sigma_3 < \dots$

Π_n^d = the d-divisible partition lattice

$= \{\pi \in \Pi_n : \forall B \in \pi \quad d \mid |B|\}$

<u>Theorem</u> [Stanley] $\mu(\Pi_n^d \cup \{\hat{0}\}) = (-1)^{n/d}$.

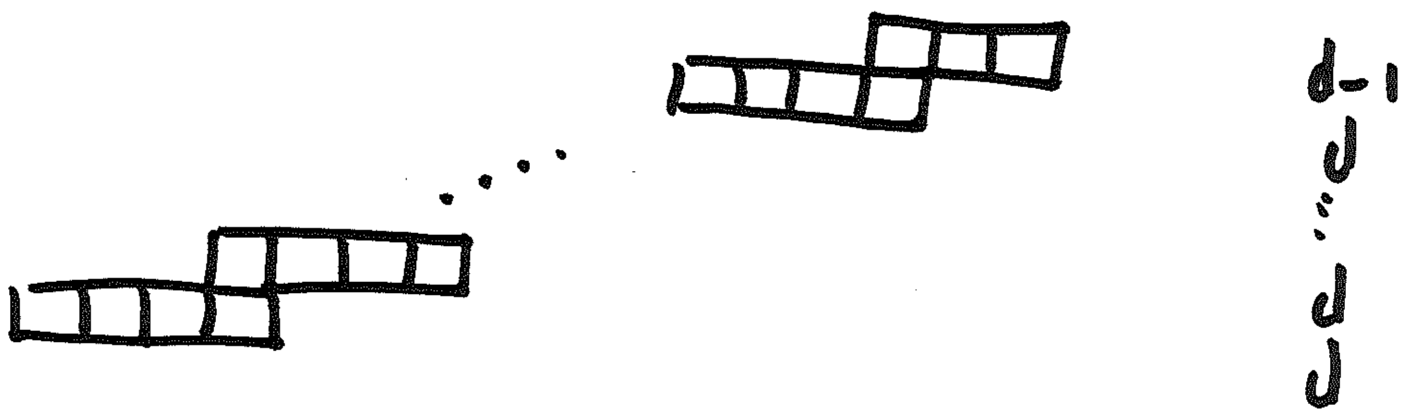
- $\# \{\alpha \in \mathfrak{S}_n : \mathrm{Des}(\alpha) = \{d, 2d, \ldots, n-d\}\ \&\ \alpha_n = n\}$

<u>Theorem</u> [Wachs] $\Pi_n^d \cup \{\hat{0}\}$ is EL-shellable.

<u>Corollary</u> $\Delta(\Pi_n^d - \{\hat{1}\})$ is a wedge of $(n/d - 2)$-dimensional spheres.

Theorem [Calderbank–Hanlon–Robinson] [Wachs]

The action on $\tilde{H}_{n/d-2}(\Delta(\Pi_n^d - \{\hat{1}\}))$ of the symmetric group $\mathfrak{S}_{n-1}$ is isomorphic to the Specht module of shape



$$
\begin{tikzcd}
\text{set partitions } \Pi_n \arrow[d, "\text{type}"'] & & \text{ordered set partitions } Q_n \arrow[ll, "\text{forgetful map}"'] \arrow[d, "\text{type}"] \\
\text{integer partitions } I_n & & \text{compositions } \mathrm{Comp}(n) \arrow[ll, "\text{type}"']
\end{tikzcd}
$$

VII

$$
\begin{tikzcd}
143|25|67 \arrow[d, mapsto] & 25-143-67 \arrow[l, mapsto] \arrow[d, mapsto] \\
\{2,2,3\} & (2,3,2) \arrow[l, mapsto]
\end{tikzcd}
$$

Poset structure

Π_n inclusion

Q_n merging adjacent blocks

$\mathrm{Comp}(n)$ adding adjacent entries $\cong B_{n-1}$

I_n adding entries

$\vec{c}$ composition of n

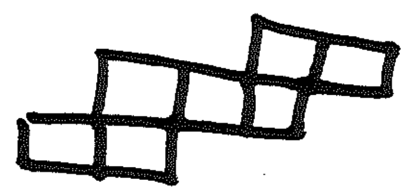
$$Q^*_{\vec{c}} = \{ \sigma \in Q_n : \text{type}(\sigma) \geq \vec{c}$$
$$\& \ n \in \text{last block of } \sigma \}$$

$$\Pi_{\vec{c}} = f(Q^*_{\vec{c}})$$

forgetful

$\beta_n^*(\vec{c}) = \# \, \alpha \in \mathfrak{S}_n$ with descent composition $\vec{c}$ and $\alpha_n = n$

$B^*(\vec{c})$ = border shape with $c_1, c_2, \ldots, c_{k-1}, c_k - 1$ boxes in each row

$\vec{c} = (2,3,3)$ $\qquad B^*(\vec{c}) =$ 

$$S^{B^*(\vec{\tau})} = \text{Specht module of shape } B^*(\vec{\tau})$$

$$\dim\left(S^{B^*(\vec{\tau})}\right) = \beta_n^*(\vec{\tau})$$

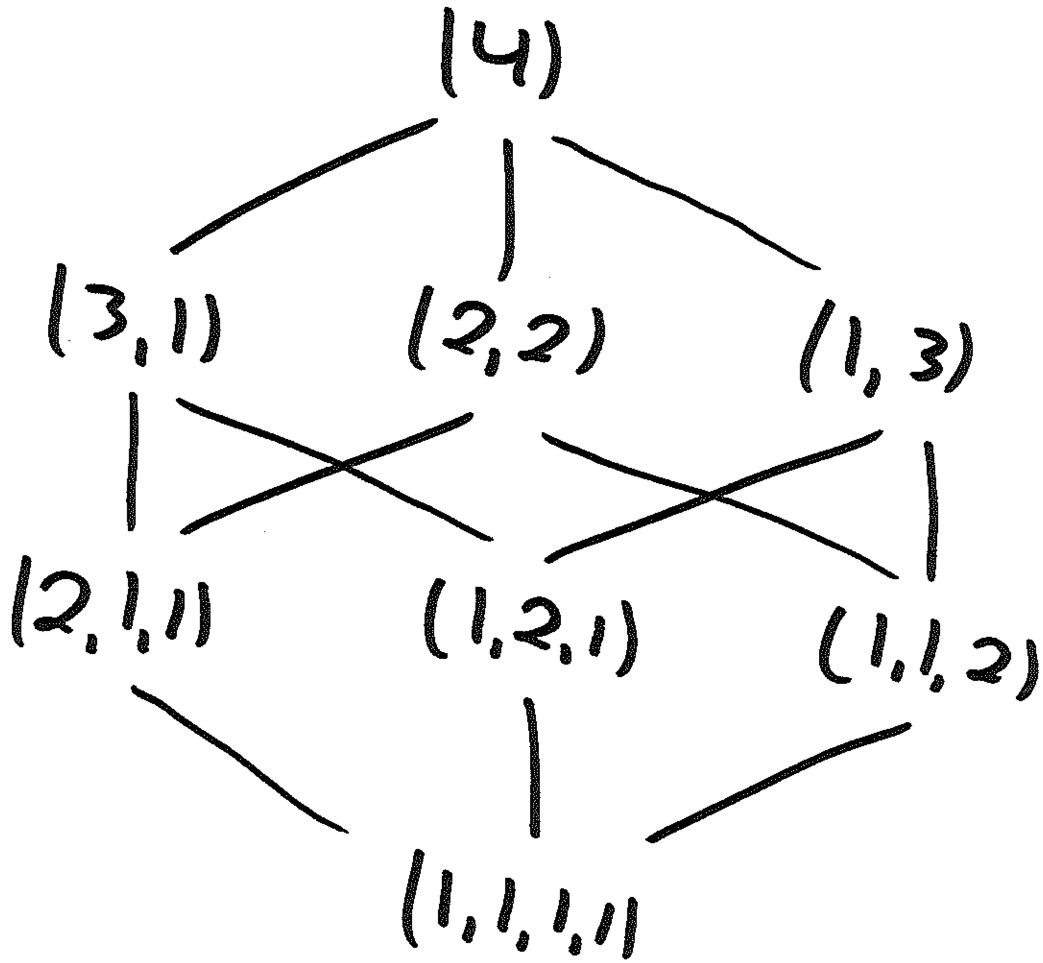
Theorem [E-Jung]

$\vec{c} = (c_1, c_2, \ldots, c_k)$ composition of n.

$\Delta(\Pi_{\vec{c}} - \{\hat{1}\})$ homotopy equivalent to a wedge of $\beta_n^*(\vec{c})$ spheres of dimension $k-2$.

$$\widetilde{H}_{k-2}\left(\Delta(\Pi_{\vec{c}} - \{\hat{1}\})\right) \cong_{\mathfrak{S}_{n-1}} S^{B^*(\vec{c})}$$

Comp(4)



$$(1+1+1, 1, 1+1, 1) \longleftrightarrow {+}{+}\,,\,,\,{+}\,,$$

Δ filter in Comp(n)

$\vec{c} \in \Delta, \ \vec{c} \leq \vec{d} \implies \vec{d} \in \Delta$

By viewing Δ upside down,
Δ is a simplicial complex.

Δ filter in $\mathrm{Comp}(n)$

$$Q^*_\Delta = \{\sigma \in Q_n : \text{type}(\sigma) \in \Delta \\ n \in \text{last block of } \sigma\}$$

$$\Pi_\Delta = f(Q^*_\Delta)$$

<u>Theorem</u> [E-H]

$$\widetilde{H}_i(\Delta(\Pi_\Delta - \{\hat{1}\})) \cong \bigoplus_{\vec{\tau} \in \Delta} H_{i-|\vec{\tau}|+1}(\mathrm{lk}_{\vec{\tau}}(\Delta)) \otimes S^{B^*(\vec{\tau})}$$

as $\mathfrak{S}_{n-1}$-modules

<u>Corollary</u> Δ k-dimensional sphere. Then $\Delta(\Pi_\Delta - \{\hat{1}\})$ only has k-dimensional reduced homology and

$$\widetilde{H}_k(\Delta(\Pi_\Delta - \{\hat{1}\})) \cong \bigoplus_{\vec{\tau} \in \Delta} S^{D^*(\vec{\tau})}$$

Corollary Δ k-dimensional ball. Then $\Delta(\Pi_\Delta - \{\hat{1}\})$ only has k-dimensional reduced homology and

$$\tilde{H}_k(\Delta(\Pi_\Delta - \{\hat{1}\})) \cong \bigoplus_{\vec{c} \in \mathrm{Int}(\Delta)} S^{B^*(\vec{c})}$$

<u>Theorem</u> [E-H] If every link $\mathrm{lk}_{\vec{c}}(\Delta)$ has a Morse matching such that the critical cells are facets, then $\Delta(\Pi_\Delta - \{\hat{1}\})$ is a wedge of spheres. Number of i-dim spheres is

$$\sum_{\vec{c} \in \Delta} \beta_n^*(\vec{c}) \cdot \tilde{\beta}_{i-|c|+1}(\mathrm{lk}_{\vec{c}}(\Delta))$$

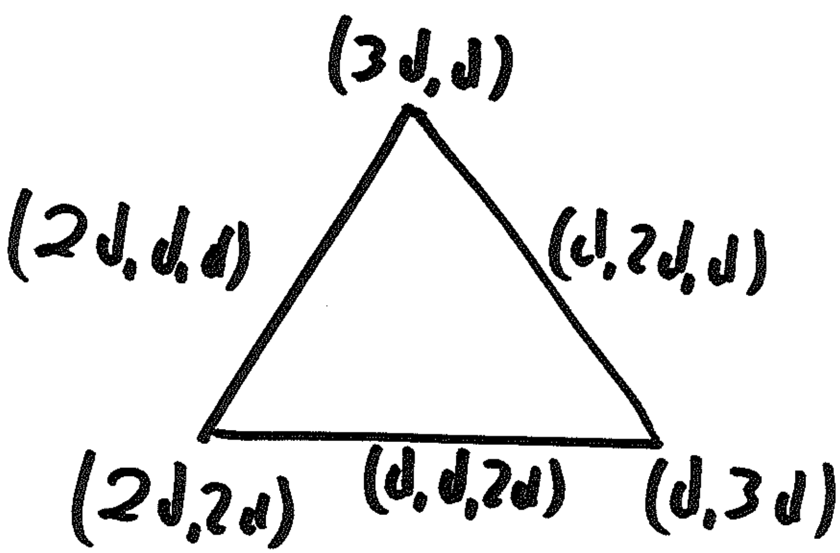
For instance, Δ non-pure shellable.

<u>Corollary</u> Δ shellable and dimension k.
Then $\Delta(\Pi_\Delta - \{\hat{1}\})$ is a wedge of k-dimensional spheres. Number of spheres is

$$\sum_{\vec{c} \in \Delta} \beta_n^*(\vec{c}) \cdot \tilde{\beta}_{k-|c|+1}(\operatorname{lk}_{\vec{c}}(\Delta))$$

Ex Π_n^d = d-divisible partition lattice

$$\Delta(\Pi_n^d - \{\hat{1}\} - \{\text{minimal elements}\})$$



$$\Delta = \{\vec{c} <^* (d, d, \ldots, d)\}$$

$$\Delta = \{ \vec{\tau} <^* (d, d, \ldots, d) \} = \partial\left(\left(\tfrac{n}{d} - 2\right) - \text{simplex} \right)$$

$$= \left(\tfrac{n}{d} - 3\right) - \text{sphere}$$

$$\widetilde{H}_{n/d-3}(\quad) \cong \bigoplus_{\vec{\tau} <^* (d, d, \ldots, d)} S^{D^*(\vec{\tau})}$$

$$\cong M^{D^*(d, d, \ldots, d)} \ominus S^{D^*(d, d, \ldots, d)}$$

Ex Λ = semigroup $\subseteq \mathbb{P}$

$\Lambda = \langle a, a+d, a+2d, \ldots \rangle$ $\gcd(a,d)=1$

Δ_n = all compositions of n with entries $\in \Lambda$

Δ_n has a Morse matching with all critical cells are facets [Clark–E]

$$\sum_{i \geq -1} \sum_{n \geq 0} \tilde{\beta}_i(\Delta_n) \cdot q^n \cdot t^{i+1} = \frac{q^a \cdot [a]_{q^d}}{1 - q^{a+d} \cdot [a-1]_{q^d} \cdot t}$$

$$\mathrm{lk}_{\vec{c}}(\Delta_n) = \Delta_{c_1} * \Delta_{c_2} * \cdots * \Delta_{c_k}$$

$$\Psi_n = [q^n] \frac{q^a \cdot [a]_{q^d}}{1 - q^{a+d} \cdot [a-1]_{q^d} \cdot t}$$

$$\widetilde{H}_i(\Delta(\Pi_{\Delta_n} - \{\hat{1}\})) \cong \bigoplus_{\vec{c} \in \Delta_n} [t^{i-|\vec{c}|+2}] \Psi_{c_1} \cdot \Psi_{c_2} \cdots \Psi_{c_k} \cdot S^{B^*(\vec{c})}$$

$d=1$ $\quad \Lambda = \{a, a+1, a+2, \ldots\}$

Billera–Myers $\Rightarrow$ Δ_n non-pure shellable

Björner–Wachs: $\Pi_{\Delta_n} \cup \{\hat{0}\}$ EL-shellable.

Questions

For which $\Delta \subseteq \mathrm{Comp}(n)$ is $\Pi_\Delta \cup \{\hat{0}\}$ EL-shellable?

Action of $\mathfrak{S}_n$?

$\Omega \subseteq \mathrm{Comp}(n)$, not a filter.
What can we say about Π_Ω?

Thank you!