A new shellability proof of an old identity of Dixon

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Combinatorial identity (Dixon 1891):

$$\sum_{s=0}^{n}(-1)^s\binom{n}{s}^3 = \begin{cases} 0 & \text{if } n \text{ is odd, and} \\ (-1)^{n/2}\binom{3n/2}{n/2,n/2,n/2}, & \text{if } n \text{ is even.} \end{cases}$$

Fun history facts:

- Hard to find original manuscript: *Messenger of Mathematics*- ended 1929
- See MacMahon’s *Combinatory Analysis* (1915)
- Zillions of ways to prove it. Ex: MacMahon’s Master Theorem!
- Special case of multiple families of identities (hypergeometric and "beyond")

Reformulate Dixon's Identity as a Shellability Exercise

$$\sum_{s=0}^{n}(-1)^s\binom{n}{s}^3 = \begin{cases} 0 & \text{if } n \text{ is odd, and} \\ (-1)^{n/2}\binom{3n/2}{n/2,n/2,n/2}, & \text{if } n \text{ is even.} \end{cases}$$

- Find a shellable simplicial complex with face numbers $f_{s-1} = \binom{n}{s}^3$ and reduced Euler Characteristic $\sum_{s=0}^{n}(-1)^{s+1}\binom{n}{s}^3$
- Betti numbers $\widetilde{\beta}_i$ count i-dimensional facets attaching along entire boundary in shelling order. Use Euler-Poincaré Relation: $\sum_{i=-1}^{d}(-1)^i f_i = \sum_{i=-1}^{d}(-1)^i \widetilde{\beta}_i$.

Story: Elkies –> Hersh/Stanton –> Hersh –> Me – - - - –> Me + Dan + Tina

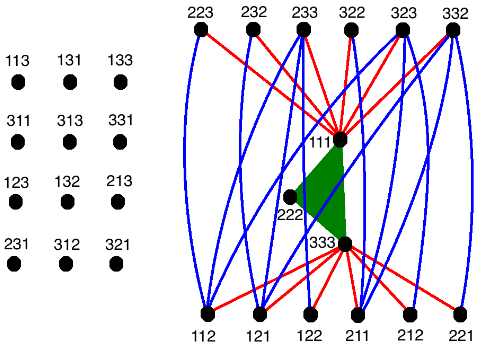
Definition (Hersh). Let $\Delta(n)$ be the simplicial complex with ground set $\{(i_\ell, j_\ell, k_\ell) \mid i_\ell, j_\ell, k_\ell \in [n]\}$ and faces given by collections $\{(i_1, j_1, k_1), \ldots, (i_s, j_s, k_s)\}$ satisfying

$$i_1 < i_2 < \cdots < i_s, \quad j_1 < j_2 < \cdots < j_s, \quad k_1 < k_2 < \cdots < k_s.$$

Example. $F = \{(1,3,3),(5,4,5)\}$ is a face of $\Delta(k)$ for $k \geq 5$, but F is a facet of $\Delta(5)$. $G = \{(1,3,3),(5,5,5)\}$ is not a facet of $\Delta(5)$ because $G \subset H$ where $H = \{(1,3,3),(4,4,4),(5,5,5)\}$.

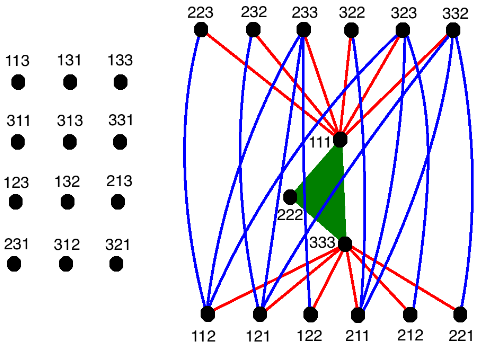
- $\Delta(n)$ has $\binom{n}{s}^3$ $(s-1)$-dimensional faces $\iff$ $f_{s-1} = \binom{n}{s}^3$.
- Facets of $\Delta(n)$ satisfy
  1. $1 \in \{i_1, j_1, k_1\}$
  2. $n \in \{i_s, j_s, k_s\}$
  3. $\min\{i_\ell - i_{\ell-1}, j_\ell - j_{\ell-1}, k_\ell - k_{\ell-1}\} = 1$ for all $\ell \in \{2, \ldots s\}$

Theorem. The lex order on the label sequences $(i_1, j_1, k_1, i_2, j_2, k_2, \ldots, i_r, j_r, k_r) \in [n]^{3r}$ of the facets of $\Delta(n)$ taken in descending dimension induces a shelling of $\Delta(n)$.



Facet $\{(1,1,2),(2,3,3)\}$ has label sequence $(1,1,2,2,3,3)$.

Proposition. The homology facets of $\Delta(n)$ are the facets $\{(i_1, j_1, k_1), \ldots, (i_r, j_r, k_r)\}$ satisfying $\max\{i_1, j_1, k_1\} > 1$ and $(i_\ell - i_{\ell-1}) + (j_\ell - j_{\ell-1}) + (k_\ell - k_{\ell-1}) > 3$ for all $\ell \in \{2, 3, \ldots, r\}$.



Example: $\{(1, 1, 2), (2, 3, 3)\}$ is a homology facet of $\Delta(3)$

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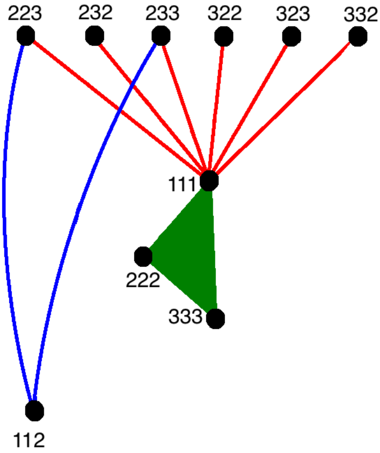

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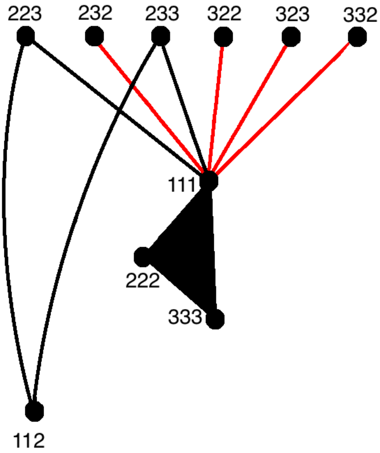

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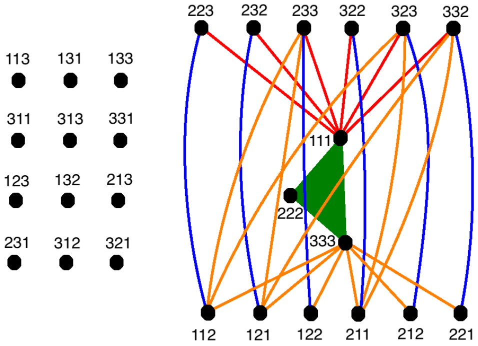

Note: 3 is odd and $\sum_{i=-1}^{1} \widetilde{\beta}_i(\Delta(3)) = 0$.

Values are homology facets with r vertices.

	$\Delta(2)$	$\Delta(3)$	$\Delta(4)$	$\Delta(5)$	$\Delta(6)$	$\Delta(7)$
$r = 1$	6	12	18	24	30	36
$r = 2$	0	12	114	396	948	1860
$r = 3$		0	6	372	3138	13704
$r = 4$			0	0	540	12240
$r = 5$				0	0	360
$r = 6$					0	0
$r = 7$						0
| A. S. |	6	0	90	0	1680	0

Bummer. Not in the OEIS...
Bummer?

- Suppose $i_1 = 1, j_1, k_1 > 1$:

$$f(x,y,z) = x \sum_{j_i=2}^{\infty} \sum_{k_i=2}^{\infty} y^{j_i^2} z^{k_i^2} = \frac{xy^2z^2}{(1-y)(1-z)}.$$

- Supppose $i_i = 1$, and $j_i = 1$:

$$h(x,y,z) = xy \sum_{k_i=2}^{\infty} z^{k_i} = \frac{xyz^2}{1-z}.$$

- So

$$P(x,y,z) = f(x,y,z)+f(y,x,z)+f(z,y,x)+h(x,y,z)+h(x,z,y)+h(z,y,x)$$

$$\iff P(x,y,z) = xyz\left(\frac{1-xyz}{(1-x)(1-y)(1-z)} - 1\right).$$

Right GF = Two GFs from Two Types of Homology Facets

Add Case (1) $\min\{i_r, j_r, k_r\} < n$, to Case (2) $(i_r, j_r, k_r) = (n, n, n)$:

$$\begin{aligned}
\chi(x,y,z) &= \sum_{r=1} P(x,y,z)^r(-1)^{r-1} + xyz\sum_{r=1} P(x,y,z)^{r-1}(-1)^{r-1} \\
&= (P(x,y,z)+xyz)\sum_{r=1}^{\infty} P(x,y,z)^{r-1}(-1)^{r-1} \\
&= \frac{xyz(1-xyz)}{(1-x)(1-y)(1-z)}\sum_{r=1}^{\infty} P(x,y,z)^{r-1}(-1)^{r-1} \\
&= \frac{xyz(1-xyz)}{(1-x)(1-y)(1-z)}\left(\frac{1}{1-xyz+\frac{xyz(1-xyz)}{(1-x)(1-y)(1-z)}}\right) \\
&= \frac{xyz}{(1-x)(1-y)(1-z)+xyz}
\end{aligned}$$

Hit it with MacMahon's Master Theorem..... get

$$[x^n y^n z^n] = \begin{cases} 0 & \text{if } n \text{ is odd, and} \\ (-1)^{n/2}\binom{3n/2}{n/2,n/2,n/2}, & \text{if } n \text{ is even.} \end{cases}$$

Related identities

Generalized $\Delta(n)$: $\Gamma_p(n)$ with vertices given by sequences in $[n]^p$ for $p \geq 1$, and faces given by collections of vertices

$$\{(i_{1,1}, \ldots, i_{1,p}), (i_{2,1}, \ldots, i_{2,p}), \ldots, (i_{r,1}, \ldots, i_{r,p})\}$$

satisfying $i_{\ell,a} \in [n]$ for all $\ell \in [r]$ and all $a \in [p]$, and $i_{\ell,a} < i_{(\ell+1),a}$ for all $\ell \in [r-1]$ and all $a \in [p]$. Then $\Gamma_p(n)$ has face numbers given by

$$f_{s-1} = \binom{n}{s}^p$$

for $0 \leq s \leq n$.

$\Gamma_1(n)$: (EC Vol 1 Exercise 1.3-f)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0, \quad n \geq 1; \tag{1}$$

$\Gamma_1(n)$ corresponds to Δ_{n-1} with labeled vertices: homotopy type of a point.

$\Gamma_2(n)$ corresponds to the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ (-1)^{n/2}\binom{n}{n/2}, & \text{if } n \text{ is even}. \end{cases} \tag{2}$$

Summary

- We have recast an old identity of Dixon as the reduced Euler characteristic of a non-pure shellable simplicial complex.
- We identified the homology facets of the shellable complex and counted them using a generating function.
- Machine for generating new identities?
- More interesting: a way to peek “under the hood" about the Master Theorem, connections between types of enumeration...

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