# Equivariant K-theory and Set Partitions

Mahir Bilen Can, Soumya Banerjee
mcan@tulane.edu, soumya@math.bgu.ac.il

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Let $G = \mathrm{SL}_{2n}(\mathbb{C})$ and $H = \mathrm{Sp}_n(\mathbb{C})$ denote its symplectic subgroup.

$$\dim G = 4n^2 - 1, \qquad \dim H = n(2n+1) = 2n^2 + n.$$

Consider the Lie algebra of $H$ as a point in the Grassmann of $n(2n+1)$ dimensional subspaces of $\mathfrak{sl}_{2n}$ and define

$$X_{G/H} := \overline{G \cdot \mathfrak{sp}_n} \subset \mathrm{Gr}(n(2n+1), \mathfrak{sl}_{2n})$$

For an arbitrary pair $H \subset G$, $X_{G/H}$ is not necessarily nonsingular. However, when $H$ is the fixed point subgroup of an involutory automorphism

$$\theta : G \to G,$$

then by a result of De Concini and Procesi (1982) this is the case.

Moreover, in this case, $X_{G/H}$ is spherical.

In Franch these varieties are called "prolongement magnifique" (wonderful compactifications).

$X_{G/H}$ has a unique closed $G$-orbit which is a partial flag variety $G/P$.

- $P$: parabolic subgroup opposite to $\theta(P)$
- $L = P \cap \theta(P)$
- $T \subset L$ a maximal torus
- $T_0 = \{t \in T : \ \theta(t) = t\}$
- $T_1 = \{t \in T : \ \theta(t) = t^{-1}\}$
- $W_G, W_H, W_L$ associated Weyl groups
- $\Phi_G, \Phi_H, \Phi_L$ the root systems of $(G, T), (H, T_0), (L, T)$

- If $p : \mathcal{X}(T) \to \mathcal{X}(T_1^0)$ is the restriction map, then

$$\Phi_{G/H} := p(\Phi_G) - \{0\}$$

is a root system, which is possibly non reduced.

- $\Delta_{G/H} = \{\alpha - \theta(\alpha) : \ \alpha \in \Delta_G - \Delta_L\}$ is a basis for $\Phi_{G/H}$.
- The little Weyl group of $G/H$ is defined as

$$W_{G/H} := N_G(T_1^0)/Z_G(T_1^0).$$

The closure $Y := \overline{T/T_0} \subset X_{G/H}$ is a toric variety and it is of the form $W_{G/H} \cdot Y_0$ where $Y_0$ is the affine toric subvariety of $Y$ associated with the positive Weyl chamber dual of $\Delta_{G/H}$.

$Y_0$ has a unique $T$-fixed point, denoted by $z_0$.

The *rank* of $G/H$ is defined to be the dimension of $T_1$. $G/H$ is called of *minimal rank* if $rank(G/H) + rank(H) = rank(G)$.

## Lemma (Brion-Joshua, Tchoudjiem)

1. *The $T$-fixed points in $X_{G/H}$ (reps. $Y$) are exactly the points $w \cdot z_0$, where $w \in W_G$, (resp. $W_H$). These fixed points are parametrized by $W_G/W_L$ (reps., $W_H/W_L \cong W_{G/L}$).*
2. *For any $\alpha \in \Phi_G^+ - \Phi_L^+$, there exists unique irreducible $T$-stable curve $C_{\alpha,z_0}$ which contains $z_0$ and on which $T$ acts through its character $\alpha$. The $T$-fixed points in $C_{\alpha,z_0}$ are precisely $z_0$ and $s_\alpha \cdot z_0$.*
3. *For any $\gamma = \alpha - \theta(\alpha) \in \Delta_{G/H}$, there exists unique irreducible $T$-stable curve which contains $z_0$ and on which $T$ acts by the character $\gamma$. The $T$-fixed points are exactly $z_0$ and $s_\alpha s_{\theta(\alpha)} \cdot z_0$.*
4. *The irreducible $T$-stable curves in $X_{G/H}$ are $W_G$-translate of the curves $C_{\alpha,z_0}$ and $C_{\gamma,z_0}$. They are all isomorphic to $\mathbb{P}^1$.*
5. *The irreducible $T$-stable curves in $Y$ are the $W_{G/H}$-translates of the curves $C_{\gamma,z_0}$.*

Using this lemma, Brion and Joshua found a description of the equivariant Chow ring of a wonderful compactification of minimal rank. We do the same with the algebraic K-theory to a finer degree by using combinatorics.

We go back to our original example $G = \mathrm{SL}_{2n}$, $H = \mathrm{Sp}_n$ for $\theta(g) = J_n(g^\top)^{-1}J_n^{-1}$, where

$$J_n = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

The maximal torus of diagonal matrices in $G$ is $\theta$-stable. The set of simple roots of $\Phi(G, T)$ relative to Borel subgroup of upper triangular matrices is then $\Delta_G = \{\alpha_1, \dots, \alpha_{2n-1}\}$. The action of $\theta$ on $\Phi$ is given by

$$\theta(\alpha_i) = \begin{cases} \alpha_i & \text{if } i \text{ is odd;} \\ -(\alpha_{i-1} + \alpha_i + \alpha_{i+1}) & \text{if } i \text{ is even.} \end{cases}$$

Furthermore,

$$P = \begin{pmatrix} * & * & * & * & \cdots & * & * & * & * \\ * & * & * & * & \cdots & * & * & * & * \\ 0 & 0 & * & * & \cdots & * & * & * & * \\ 0 & 0 & * & * & \cdots & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & * & * & * & * \\ 0 & 0 & 0 & 0 & \cdots & * & * & * & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & * \end{pmatrix}$$

and

$$L = P \cap \theta(P) \simeq \mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2.$$

The set of restricted simple roots is

$$\Delta_{G/H} = \{\overline{\alpha_i} := \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} : \ i \in \{2, 4, \ldots, 2n\}\}$$

and the associated little Weyl group $W_{G/H}$ is isomorphic to $S_n$. Thus, the $T$-fixed points of $Y$ are indexed by $S_n$ and there exists a $T$-invariant curve $C_{\gamma'}$ between two fixed points if and only if there is a $T$-invariant curve $C_\gamma$ connecting $z_0$ to $s_{\alpha_i} s_{\theta(\alpha_i)} \cdot z_0$ and $C_{\gamma'}$ is the $w$ translate of $C_\gamma$ for some $w \in S_n$.

Secretly, this is giving us the moment polytope of the toric variety $Y = W_{G/H} \cdot Y_0$, which is the permutahedron of $S_n$ in this example.

We are going to compute the equivariant K-theory of the toric variety associated with the permutahedron.

Let $\mathcal{P}^*(n)$ denote the dual of the permutahedron. The face lattice $\mathcal{L}^*(n)$ of $\mathcal{P}^*(n)$ is isomorphic to the lattice of "ordered set partitions" with respect to the reverse refinement ordering.

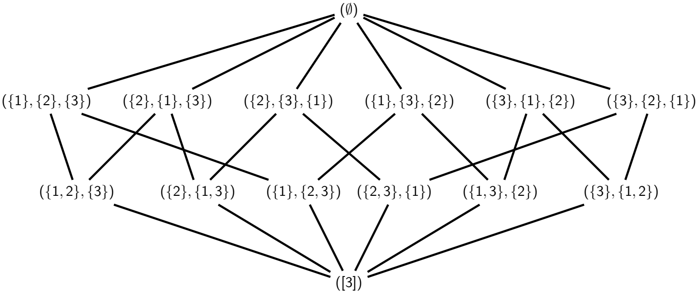


Figure: The Hasse diagram of $\mathcal{L}^*(3)$.

For each $\sigma \in S_n$, we let $\{\{\sigma_1\}, \{\sigma_2\}, \ldots, \{\sigma_n\}\}$ denote the associated ordered set partition. For each $i \in \{1, \ldots, n-1\}$ let $x_{\sigma_i \sigma_{i+1}}$ be a variable and let

$$K_{T,0}(U_\sigma) := \mathbb{Z}[x^{\pm}_{\sigma_1\sigma_2}, \ldots, x^{\pm}_{\sigma_{n-1}\sigma_n}]$$

be the Laurent polynomial ring.
Also, if $\tau \in S_n$ is another permutation, then set

$$K_{T,0}(U_{\sigma\wedge\tau}) := \mathbb{Z}[x^{\pm}_{\sigma_{i_1}\sigma_{i_1+1}}, \ldots, x^{\pm}_{\sigma_{i_r}\sigma_{i_r+1}}]$$

where

$$\{x_{\sigma_{i_1}\sigma_{i_1+1}}, \ldots, x_{\sigma_{i_r}\sigma_{i_r+1}}\} = \{x_{\sigma_1\sigma_2}, \ldots, x_{\sigma_{n-1}\sigma_n}\} \cap \{x_{\tau_1\tau_2}, \ldots, x_{\tau_{n-1}\tau_n}\}.$$

In this case, define

$$B(\sigma,\tau) := \{x_{\sigma_1\sigma_2},\dots,x_{\sigma_{n-1}\sigma_n}\} - \{x_{\sigma_{i_1}\sigma_{i_1+1}},\dots,x_{\sigma_{i_r}\sigma_{i_r+1}}\}$$

**Theorem (S. Banerjee, M.B.C. 2013)**

*The $T$-equivariant $K$-theory of the toric variety $Y$ is the subring of $U = \bigoplus_{\sigma\in S_n}\mathbb{Z}[x^{\pm}_{\sigma_1\sigma_2},\dots,x^{\pm}_{\sigma_{n-1}\sigma_n}]$ consisting of elements $(g_\sigma)_{\sigma\in S_n}\in U$ such that $g_\sigma - g_\tau = 0$ $(\sigma,\tau\in S_n)$ whenever variables from the set $B(\sigma,\tau)$ are set to zero.*

## Theorem (S. Banerjee, M.B.C. 2013)

*The T-equivariant K-theory of the wonderful compactification $X_{G/H}$ is the subring of the ring of invariants $U^{S_2 \wr S_n} = \bigoplus_{\sigma \in S_n} \mathbb{Z}[x^{\pm}_{\sigma_1\sigma_2}, \ldots, x^{\pm}_{\sigma_{n-1}\sigma_n}]^{S_2 \wr S_n}$ consisting of elements $(g_\sigma)_{\sigma \in S_n} \in U^{S_2 \wr S_n}$ such that $g_\sigma - g_\tau = 0$ $(\sigma, \tau \in S_n)$ whenever variables from the set $B(\sigma, \tau)$ are set to zero.*

REMARK: So far what we have computed is only $K_{T,0}(Y)$ and $K_{T,0}(X_{G/H})$. To compute higher K-groups it is enough to tensor with the higher K-groups of the underlying field.

# "Just in case" slides:

- $G$: semisimple algebraic group of adjoint type
- $\theta$: automorphism of order 2 on $G$
- $H$: fixed subgroup $G^\theta = \{g \in G : \ \theta(g) = g\}$

**Theorem (De Concini-Procesi)**

*There exists unique minimal smooth projective $G$-variety $X$ having a dense open $G$-orbit isomorphic to $G/H$ and satisfying*

- *$X - G/H$ is a union $\sqcup_{\overline{\alpha} \in \overline{\Delta_1}} X_{\overline{\alpha}}$ of finitely many $G$-stable, smooth boundary divisors $X_{\overline{\alpha}}$.*
- *For any $I \subset \overline{\Delta_1}$, the intersection $\cap_{\overline{\alpha} \in I} X_{\overline{\alpha}}$ is transversal and any $G$-orbit closure in $X$ is of this form.*
- *There exists unique closed $G$-orbit which is isomorphic to $G/P$ for the parabolic $P$ determined by $\overline{\Delta_1}$.*

# “Just in case” slides:

“Irreducible” minimal rank symmetric varieties:

- $G \times G/\Delta G$
- $\mathrm{PSL}_{2n}/\mathrm{PSp}_n$
- $\mathrm{PSO}_{2n}/\mathrm{PSO}_{2n-1}$
- $\mathrm{E}_6/\mathrm{F}_4$

# "Just in case" slides:

**Theorem (S. Banerjee, M.B.C. 2013)**

*The $T$-equivariant K-theory $K_{T,0}(X)$ of minimal rank wonderful compactification $X_{G/H}$ is isomorphic to the space of tuples $(f_{w\cdot z_0}) \in \bigoplus_{w\in W_G/W_L} R(T)_{w\cdot z_0}$ such that*

$$f_{w\cdot z_0} - f_{w'\cdot z_0} = 0 \text{ in } \frac{\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]}{((x_1 - 1)(x_2 - 1))}$$

*whenever $w \cdot z_0$ and $w' \cdot z_0$ are joined by an irreducible $T$-curve.*

More generally, suppose we have the following setup:

- $G$: reductive group
- $B \subseteq G$ a Borel subgroup
- $X$: a $G$-variety with finitely many $B$-orbits.

$X$ is called spherical.

# "Just in case" slides:

**Example**

$X$ can be one of the following varieties

- a toric variety
- a partial flag variety
- a symmetric space or any of its equivariant compactification
- a linear algebraic monoid

Our objective here is to describe the equivariant K-theory of a specific example.

Slaughtering K-theory:

- $\mathcal{C}$: a small category;
- $B\mathcal{C}$: the classifying complex of $\mathcal{C}$, which, by definition, is the topological realization of the simplicial complex whose simplicies are chains of morphisms.

**Definition**

• $n$th K-group of $\mathcal{C}$ is the $n$th homotopy group of $B\mathcal{C}$.
• If $X$ is a $G$-variety, then its $n$th $G$-equivariant K-group is the $n$th K-group of the (small) category of $G$-equivariant vector bundles on $X$.

How on earth anyone can compute anything using this definition?

**Theorem (Vezzosi-Vistoli '03)**

*Suppose $D$ is a diagonalizable group acting on a smooth proper scheme $X$ defined over a perfect field; denote by $T$ the toral component of $D$, that is the maximal subtorus contained in $D$. Then the restriction homomorphism on $K$-groups $K_{D,*}(X) \to K_{D,*}(X^T)$ is injective, and its image equals the intersection of all images of the restriction homomorphisms $K_{D,*}(X^S) \to K_{D,*}(X^T)$ for all subtori $S \subset T$ of codimension 1.*

Therefore, for a spherical $G$-variety $X$, we need to analyze $X^S$. when $S$ is a codimension one subtorus of $T$.

# "Just in case" slides:

It turns out that for a smooth complete spherical variety, $X^S$ is a spherical $C_G(S)$-variety. There are two possibilities:

i) $S$ is regular. Then the action of $T = C_G(S)$ on $X^S$ factors through $T/S \simeq \mathbb{C}^*$. Hence, irreducible components of $X^S$ are either points, or $\mathbb{P}^1$'s.

ii) $S$ is singular. Then the semisimple rank of $C_G(S)$ is 1, hence $X^S$ is a spherical $\mathrm{SL}_2$- or $\mathrm{PSL}_2$-variety.

In the latter case, an irreducible component of $X^S$ is either a copy of $\mathbb{P}^2$ on which $\mathrm{SL}_2$ acts by its 3-dimensional representation, or it is a Hirzebruch surface

$$F_n \simeq \mathrm{SL}_2 \times_{B^-} \mathbb{P}^1$$

on which $\mathrm{SL}_2$ acts on the first factor.

$$\left( \text{Here, } B^- \text{ acts on } \mathbb{P}^1 \text{ via } \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \cdot [x_0 : x_1] = [x_0, a^n x_1]. \right)$$

# “Just in case” slides:

Thus, each irreducible component $Y$ of $X^S$ is a toric variety. The $T$-equivariant K-theory of such a component reduces to the computation of $T^b$-equivariant K-theory for a much more smaller torus $T^b$, for which

$$T/S \hookrightarrow T^b.$$

# "Just in case" slides:

**Theorem (Banerjee, C., 2013)**

*Let $G$ be any connected reductive group and $X$ be a smooth complete spherical $G$ variety. Let $T$ denote the maximal torus of $G$ and $X^T$ denote the finite set of fixed points of $T$ in $X$. For each $\gamma \in X^T$ we denote by $R(T)_\gamma$ a copy of the representation ring of $T$, and set $T^b = \mathbb{C}^* \times \mathbb{C}^*$. Then there is an injective map*

$$\iota : K_{T,0}(X) \hookrightarrow K_{T,0}(X^T) \simeq \prod_{\gamma \in X^T} R(T)_\gamma \qquad (*)$$

*of $R(T)$ modules such that an element $(z_\gamma)$ from the right hand side of $(*)$ lies in the image of $\iota$ if and only if the following holds:*

# "Just in case" slides:

## Theorem (Continued)

1. $z_{\gamma_1} - z_{\gamma_2} = 0$ *in* $\frac{\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]}{((x_1-1)(x_2-1))} = K_{T,0}(X^S)$ *whenever there is a* $T$ *invariant curve joining points* $\gamma_1$ *and* $\gamma_2$ *isomorphic to* $X^S$ *for some codimension one subtorus* $S \subset T$.
2. $z_{\gamma_1} - z_{\gamma_2} + z_{\gamma_3} = 0$ *in* $\frac{\mathbb{Z}[x_\rho^{\pm 1}]}{(\prod_{\rho \in \Psi}(x_\rho - 1))} \otimes_{R(T^b)} R(T) = K_{T,0}(X^S)$ *whenever there is a* $T$ *invariant projective plane* $\mathbb{P}^2$ *containing torus fixed points* $\gamma_1, \gamma_2,$ *and* $\gamma_3$ *isomorphic to* $X^S$ *for some codimension one subtorus* $S \subset T$.
3. $z_{\gamma_1} - z_{\gamma_2} + z_{\gamma_3} - z_{\gamma_4} = 0$ *in* $\frac{\mathbb{Z}[x_\rho^{\pm 1}]}{(\prod_{\rho \in \Psi}(x_\rho - 1))} \otimes_{R(T^b)} R(T) = K_{T,0}(X^S)$ *whenever there is a* $T$ *invariant Hirzebruch surface* $F_n$ *containing torus fixed points* $\gamma_1, \gamma_2, \gamma_3,$ *and* $\gamma_4$ *and isomorphic to* $X^S$ *for some codimension one subtorus* $S \subset T$.

# "Just in case" slides:

## Theorem (Continued)

*In the cases of (ii) and (iii), the set $\Psi$ is a subset of $\Delta_1$ listed in Figure 2 satisfying the condition that not all of its elements are contained in a maximal cone of $\Delta_1$.*

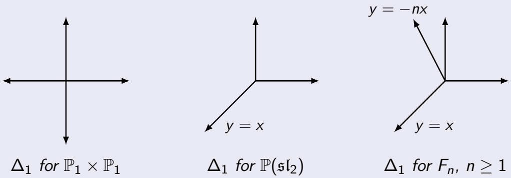


Figure: Fans of the irreducible components $Y \subset X^S$

**Theorem (Continued)**

*Moreover, the Weyl group $W$ of $G$ acts on the torus fixed points and hence it induces an action on $\prod_{\gamma \in X^T} R(T_\gamma)$. The $G$-equivariant $K$-theory of $X$ is given by the space of invariants on the right hand side of $(*)$. In other words,*

$$K_{G,0}(X) = \ image(\iota) \cap \prod_{\gamma \in X^T} (R(T)_\gamma)^W.$$